UDC 518.1

THE APPLICATION OF THE BOUNDARY INTEGRAL EQUATION METHOD TO NUMERICAL SOLUTION OF DIRICHLET'S PROBLEM IN DOMAINS WITH CORNER POINTS

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Dirichlet's problem in a domain with corner points is reduced to a boundary integral equation. In order to solve this problem numerically, we propose a method with exponential rate of convergence. An approach for computing the normal derivative of the solution to the problem is discussed. Some estimates for the number of arithmetic operations needed are given.

1. Problem statement. Let Ω be a bounded domain in $\mathbf{R}^2$ such that its boundary Γ is a closed nonoverlapping curve which admits the following parametric representations:

$$\Gamma = \big\{ x = x(s) = \big(x_1(s), x_2(s)\big), \quad s \in [0,T], \quad x(0) = x(T) \big\}$$

Here s is the natural parameter (the arc length).

We shall assume in what follows that

$$\Gamma = \bigcup_{j=0}^{J-1} \Gamma_j$$

where Γ_j is the rectilinear segment that connects the corner points P_j and P_{j+1} (it is also assumed that $P_J = P_0$). Denote by α_j the interior angle of the domain Ω at P_j and suppose $0 < \alpha_j < 2\pi$ for all j.

Let us consider the following boundary value problem:

$$\begin{aligned} &\Delta U(x) = 0\,, \qquad x \in \Omega \\ &U(x) = F(x)\,, \quad x \in \Gamma \end{aligned} \tag{1}$$

Here the continuous function F is infinitely differentiable everywhere on Γ with the exception of corner points where the singularities $(x - P_j)^\Theta$, $0 < \Theta < 1$, are allowed.

We shall search for a solution to problem (1) in the form of the double layer potential

$$U(x) = \frac{1}{2\pi} \int\limits_\Gamma \Phi(y)\, \frac{\partial}{\partial n_y} \ln|x-y|\, dl_y$$

with the unknown distribution density Φ that satisfies the following boundary integral equation [2]:

$$\Phi(x) + \frac{1}{\pi} \int\limits_\Gamma \Phi(y)\, \frac{\partial}{\partial n_y} \ln|x-y|\, dl_y = 2F(x)\,, \quad x \in \Gamma \setminus \bigcup_{j=0}^{J-1} \Gamma_j$$

Taking into account the above parametrization of the curve Γ, denote

$$\varphi(s) = \Phi(x(s))\,, \quad f(s) = 2F(x(s))$$

In this notation, the boundary integral equation takes the form

$$\varphi(s) + \int\limits_0^T K(s,t)\, \varphi(t)\, dt = f(s)\,, \quad s \in [0,T]\,, \quad x(s) \neq P_j \;\; \forall\, j = 0, \ldots, J-1 \tag{2}$$

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where

$$K(s,t)=\begin{cases}\dfrac{1}{\pi}\cdot\dfrac{x_1'(t)\big(x_2(s)-x_2(t)\big)-x_2'(t)\big(x_1(s)-x_1(t)\big)}{\big(x_1(s)-x_1(t)\big)^2+\big(x_2(s)-x_2(t)\big)^2}, & t\neq s\\ \dfrac{1}{2\pi}\cdot\dfrac{x_1'(t)x_2''(t)-x_2'(t)x_1''(t)}{\big(x_1'(t)\big)^2+\big(x_2'(t)\big)^2}, & t=s\end{cases}$$

Further, we relate a set of numbers $\{s_j\}$, $j=0,\dots,J$, to the corner points P_j so that

$$x(s_j)=P_j\,,\quad j=0,\dots,J$$
$$0=s_0<s_1<\ldots<s_{J-1}<s_J=T$$

In order to solve equation (2) numerically, we shall use the quadrature method according to which the integral in the left-hand side of (2) is replaced by a quadrature sum. As a result, a system of linear algebraic equations is obtained. However, this approach being applied to equation (2) leads to some difficulties associated with the fact that the kernel discontinues at the corner points. To avoid these difficulties, we rewrite equation (2) in the equivalent form

$$2\,\varphi(s)+\int\limits_0^T K(s,t)\big(\varphi(t)-\varphi(s)\big)\,dt=f(s),\quad s\in[0,T] \tag{3}$$

The main advantage of this representation consists in an increase of smoothness of the integrand for $t=s$.

2. Numerical solution of the integral equation. A composite quadrature formula was constructed in [1] to approximate the integral equation (3) on the solution with exponential accuracy with respect to the number of nodes.

The segment $[0,T]$ is subdivided into a finite number of subsegments that decrease in length when approaching the corner points. The endpoints of these subsegments in a neighborhood of each corner point s_j are specified by the formula $s_j+0.5\,(s_{j+1}-s_j)\cdot\Theta_j^k$ (or $s_j-0.5\,(s_j-s_{j-1})\cdot\Theta_j^k$), where $0<\Theta_j<1$, $k=0,1,2,\dots,N$. Here N is a natural number that characterize the condensation of this grid. On each of the subsegments, we use a Gaussian quadrature of the same order of accuracy with $n_{j,k}$ nodes such that

$$n_{j,k}\geqslant\left[\frac{\lambda_j(N-k)\ln(1+\Theta_j)+\ln N}{4\Theta_j(1+\Theta_j)^{-1}}\right]+1 \tag{4}$$

Here $0<\lambda_j<1$ is a number that depends on f and on the geometry of the domain and characterizes the singularity of the solution to equation (3) at a corner point:

$$\big|\varphi(s)-\varphi(s_j)\big|\leqslant\text{const}\cdot\big|x(s)-x(s_j)\big|^{\lambda_j}$$

Denoting

$$n=2\sum_{j=0}^{J-1}\sum_{k=1}^{N}n_{j,k}$$

we obtain the quadrature formula

$$S_n(s,\varphi)=\sum_{j=1}^{n}A_j^{(n)}K\left(s,t_j^{(n)}\right)\left(\varphi(t_j^{(n)})-\varphi(s)\right) \tag{5}$$

The following theorem is valid [1].

Theorem 1. *Let φ be a solution to equation (3). Then, for any natural number n a quadrature formula $S_n(\varphi)$ can be constructed such that*

$$\max_{s\in[0,T]}\left|\int\limits_0^T K(s,t)\big(\varphi(t)-\varphi(s)\big)\,dt-S_n(\varphi)\right|\leqslant b\cdot\exp(-c\sqrt{n})$$

where the constants are strictly positive and do not depend on a choice of n.

As a result, we obtain the system of linear algebraic equations

$$2\,\Phi_i^{(n)}+\sum_{j=1}^{n}A_j^{(n)}K\left(t_i^{(n)},t_j^{(n)}\right)\left(\Phi_j^{(n)}-\Phi_i^{(n)}\right)=f\left(t_i^{(n)}\right),\quad i=1,\dots,n \tag{6}$$

that approximates equation (3) on the solution.

Let us consider the equation

$$\varphi_n + K_n \varphi_n = f \tag{7}$$

where K_n is a linear bounded operator in the space of continuous T-periodic functions such that

$$(K_n\, v)\,(s) = v(s) + S_n(s, v)$$

for an arbitrary function v from this space.

The procedure of solving equation (7) is reduced to solving system (6), since $\Phi_i^{(n)} = \varphi_n\left(t_i^{(n)}\right)$. On the other hand, the function

$$\varphi_n(s) = \left(f(s) - \sum_{j=1}^{n} A_j^{(n)} K\left(s, t_j^{(n)}\right) \Phi_j^{(n)}\right)\left(2 - \sum_{j=1}^{n} A_j^{(n)} K\left(s, t_j^{(n)}\right)\right)^{-1} \tag{8}$$

is a solution to equation (7). It can be proved that, for sufficiently large values of n, representation (8) is realizable for all $s \in [0, T]$.

Thus, system (6) is solvable if there exists a sequence of operators $\left\{(I + K_n)^{-1}\right\}$ bounded uniformly in n.

The main result obtained in [1] is formulated as follows:

Theorem 2. *There is an integer $n_1 > 0$ such that for any integer $n > n_1$ the operator $(I + K_n)^{-1}$ exists and the inequality*

$$\left\|(I + K_n)^{-1}\right\|_C \leqslant \mathrm{const}_1$$

holds. The equation

$$\varphi_n + K_n \varphi_n = f$$

has a unique solution φ_n for which the estimate

$$\|\varphi - \varphi_n\|_C \leqslant \mathrm{const}_2 \cdot \exp\left(-c\sqrt{n}\right)$$

is valid, where φ is a solution to boundary integral equation (3) *and the constants are strictly positive and do not depend on a choice of n.*

3. Numerical solution of Dirichlet's problem. Let us study the question of numerical solution of the original boundary value problem (1) on the basis of the above approximate solution to boundary integral equation (3).

Taking into account the above parametrization of the curve Γ, the solution to problem (1) can be written down at an arbitrary interior point $(x_1,\, x_2)$ of the domain Ω as

$$U(x) = \frac{1}{2}\int_0^T K(x_1, x_2, t)\,\varphi(t)\,dt \tag{9}$$

where

$$K(x_1, x_2, t) = \frac{1}{\pi} \cdot \frac{x_1'(t)\big(x_2 - x_2(t)\big) - x_2'(t)\big(x_1 - x_1(t)\big)}{\big(x_1 - x_1(t)\big)^2 + \big(x_2 - x_2(t)\big)^2}$$

It is reasonable to use the quadrature formula (5) for the approximate evaluation of integral (9). Since

$$\int_0^T K(x_1, x_2, t)\,\varphi(t)\,dt = \int_0^T K(x_1, x_2, t)\,\varphi_n(t)\,dt + O\left(e^{-c\sqrt{n}}\right)$$

we adopt the function

$$U_n(x) = \frac{1}{2}\sum_{j=1}^{n} A_j^{(n)} K\left(x_1, x_2, t_j^{(n)}\right) \Phi_j^{(n)}$$

as an approximate solution to problem (1).

Let us estimate an error of this representation of the solution.

Theorem 3. *There exists a number $Q > 1$ such that for each $x \in \Omega$ the inequality*

$$\big|U(x) - U_n(x)\big| \leqslant c(x) \cdot \big(Q(x)\big)^{-c_1\sqrt{n}}$$

holds, where $0 < c(x) \leqslant c_2/r(x)$, $1 < Q(x) \leqslant 1+c_2r(x) \leqslant Q$, *and* $r(x)$ *is the distance from the point* x *to the curve* Γ. *Here the constants* c_1 *and* c_2 *are strictly positive and do not depend on a choice of* n.

Proof. It is necessary to estimate the error $R(x)$ of the quadrature

$$\int\limits_0^T K(x_1,x_2,t)\,\varphi_n(t)\,dt = \sum_{j=1}^n A_j^{(n)} K\left(x_1,x_2,t_j^{(n)}\right)\Phi_j^{(n)} + R(x)$$

for each $x \in \Omega$. We shall follow the technique proposed in [1] for proving Theorem 1.

For each j we first estimate the errors $R_j(x)$:

$$\int\limits_{s_j}^{s_{j+1/2}} K(x_1,x_2,t)\,\varphi_n(t)\,dt = \sum_{t_j^{(n)}\in[s_j,s_{j+1/2}]} A_j^{(n)} K\left(x_1,x_2,t_j^{(n)}\right)\Phi_j^{(n)} + R_j(x)$$

Here

$$s_{j+1/2} = s_j + 0.5\,(s_{j+1} - s_j)$$

Without loss of generality, we assume that

$$x(s_j) = (0,0)\,, \quad x(s_{j+1/2}) = (1,0)$$

The segment $[s_j, s_{j+1/2}]$ is subdivided into elementary subsegments by the following $N+1$ points:

$$s_j < t_N < \ldots < t_1 < t_0 = s_{j+1/2}$$

Here $t_k = s_j + (1+\Theta_j)^{-k}$, $k = 0, \ldots, N$, $0 < \Theta_j < 1$, and $N = O\big(\sqrt{n}\big)$.

It follows from [1] that for each $k = 1, \ldots, N$ the above-constructed function $\varphi_n(t)$ admits an analytic continuation from the segment $[t_k, t_{k-1}]$ into the circle on the complex plane with its center at the point $\big(0.5\,(t_{k-1}+t_k), 0\big)$ and its radius

$$r_k = 0.5\left(1 - (1+\Theta_j)^{-1}\right)(1+\Theta_j)^{2-k}$$

Note that this continuation is bounded by a constant independent of k.

The Gaussian quadrature with $n_{j,k}$ nodes specified by (4) is constructed on each elementary segment $[t_k, t_{k-1}]$. Let us assume that the integrand admits an analytic continuation from the segment $[t_k, t_{k-1}]$ of the real axis into the ellipse on the complex plane with its focuses at the points $(t_k, 0)$ and $(t_{k-1}, 0)$ such that it passes through the point $\big(0.5\,(t_{k-1}+t_k) - r_k, 0\big)$. Let us further assume that this continuation is bounded within this ellipse by $O\left((1+\Theta_j)^{k(1-\lambda_j)}\right)$. Then, the error of the elementary quadrature is estimated from above as follows:

$$R_k = \mathrm{const}\cdot(1+\Theta_j)^{-k\lambda_j} Q_k^{-2n_{j,k}}$$

Here Q_k is the sum of semiaxes of the second ellipse constructed from the first one by the mapping

$$z \to \frac{2z - (t_{k-1}+t_k)}{t_{k-1} - t_k}$$

Suppose the point x is at a distance of at least $2(1+\Theta_j)^{k(\lambda_j-1)}$ from the segment $[t_{k-1}, t_k]$. Then, the integrand $K(x_1,x_2,t)\,\varphi_n(t)$ admits an analytic continuation into the ellipse with its focuses at the points $(t_k, 0)$ and $(t_{k-1}, 0)$ such that it passes through the point

$$\Big(0.5\,(t_{k-1}+t_k), (1+\Theta_j)^{k(\lambda_j-1)}\Big)$$

(this continuation is bounded within the above ellipse by $\mathrm{const}\cdot(1+\Theta_j)^{k(1-\lambda_j)}$).

Hence, the following estimate holds

$$\left|\int\limits_{t_k}^{t_{k-1}} K(x_1,x_2,t)\,\varphi_n(t)\,dt - \sum_{t_j^{(n)}\in[t_k,t_{k-1}]} A_j^{(n)} K\left(x_1,x_2,t_j^{(n)}\right)\Phi_j^{(n)}\right| = R_{j,k}(x) \leqslant \mathrm{const}\cdot(1+\Theta_j)^{-k\lambda_j}\big(Q_k(x)\big)^{-2n_{j,k}}$$

Here $Q_k(x) > Q_k$.

Thus if the point x is at a sufficient distance from all the elementary segments of the curve Γ (these segments specify the composite quadrature formula (5)), then the choice of $n_{j,k}$ ensures that

$$\bigl|R(x)\bigr| \leqslant \text{const} \cdot e^{-c\sqrt{n}}$$

Let us consider the case when the point x approaches the boundary.

Suppose the point x is at a distance of $2d$ from the segment $[t_k, t_{k-1}]$, where $0 < d < (1+\Theta_j)^{k(\lambda_j-1)}$. By repeating the above reasoning, then, we obtain the estimate

$$|R_{j,k}| \leqslant \text{const} \cdot \frac{(1+\Theta_j)^k}{d} \bigl(Q_k(d)\bigr)^{-2n_{jk}}, \quad 1 < Q_k(d) < 1 + \text{const} \cdot d$$

since the sum of semiaxes of the ellipse with focuses $(t_k, 0)$ and $(t_{k-1}, 0)$ (into which the integrand can be analytically continued) tends to $(t_{k-1} - t_k)/2$ with decreasing d.

This estimate proves the theorem.

Thus, the above method proposed for solving problem (1) cannot be accepted as workable, since its usage does not guarantee the proximity of the exact and approximate solutions at any interior point of the domain Ω. However, this method can easily be modified.

Let us consider the case when $d < (1+\Theta_j)^{k(\lambda_j-1)}$ and replace the function $\varphi_n(t)$ on the segment $[t_k, t_{k-1}]$ by the polynomial $L_{2n_{j,k}}(t)$ that interpolates $\varphi_n(t)$ in the following points:

$$\frac{t_k + t_{k-1}}{2} + \frac{t_{k-1} - t_k}{2} \cos\left(\frac{\pi(2m-1)}{4n_{j,k}}\right), \quad m = 1, \ldots, 2n_{j,k}$$

Since $n_{j,k} = O(\sqrt{n})$, the number of arithmetic operations to compute $\varphi_n(t)$ at these points is $O(n\sqrt{n})$.

The estimate

$$\max_{[t_k, t_{k-1}]} \bigl|\varphi_n(t) - L_{2n_{j,k}}(t)\bigr| \leqslant \text{const} \cdot Q_k^{-2n_{j,k}}, \quad Q_k > 1$$

is valid, where Q_k was specified in the proof of Theorem 3. Hence,

$$\int\limits_{t_k}^{t_{k-1}} K(x_1, x_2, t)\, \varphi_n(t)\, dt = \int\limits_{t_k}^{t_{k-1}} K(x_1, x_2, t)\, L_{2n_{j,k}}(t)\, dt + O\left(e^{-c\sqrt{n}}\right)$$

It is sufficient to evaluate the integral in the right-hand side of this representation with an accuracy $O\left(e^{-c\sqrt{n}}\right)$. To do this, a standard adaptive quadrature may be used. Let us notice the following fact: consider the segments $[t_k, t_{k-1}]$ on $[0,1]$ such that a fixed point $x \in \Omega$ is at a distance of at most $2(1+\Theta)^{k(\lambda_j-1)}$ from each of them; then, their number is finite and depends on a position of the point x, but does not depend on n.

Now we consider the problem of computing an approximate solution to (1) at m interior points of the domain Ω with an accuracy $\varepsilon > 0$. Since a system of linear algebraic equations with $n \sim \ln^2 1/\varepsilon$ unknowns should be solved to compute an approximate solution of integral equation (3), the number of overall arithmetic operations is $O\bigl(\ln^6 1/\varepsilon + m \ln^3 1/\varepsilon\bigr)$.

This estimate is somewhat excessive, since the approximate solution is determined in $O(n)$ (but not $O(n\sqrt{n})$) operations if the point x is at a sufficient distance from the boundary.

4. Computing the normal derivative of a solution to Dirichlet's problem. The problem of computing the normal derivative of a solution to the original boundary value problem at a boundary point of the domain is much more complicated than the computation of this solution at a given point of the domain.

Let U be a solution to problem (1) and consider the problem of computing $\partial U(x)/\partial n_x$ at an arbitrary boundary point x^0 under the condition that this point is not a corner one.

Suppose $x^0 = \bigl(x_1(\xi_0), (x_2(\xi_0)\bigr)$ and $\xi_0 \in (s_j, s_{j+1/2}]$ for some $j = 0, \ldots, J-1$. Let us assume that $x_2(s) = 0$ for $s \in [s_j, s_{j+1/2}]$ and $x^0 = (0,0)$ up to a linear change of coordinates and that for some integer $n > 0$ which satisfies the hypotheses of Theorem 2 we computed a solution $\bigl\{\Phi_i^{(n)}\bigr\}$, $i = 1, \ldots, n$, to the approximating linear system (6) (hence, we can compute the solution φ_n to equation (7) (i.e., the approximate solution to equation (3)) at any point of the domain).

Let us choose numbers $\delta_1, \delta_2 > 0$ such that the segment $[\xi_0 - \delta_1, \xi_0 + \delta_2]$ is formed by the elementary segment of the composite quadrature formula (5) the point ξ_0 belongs to and by the two adjacent elementary segments.

The boundary segment that corresponds to $[\xi_0 - \delta_1,\, \xi_0 + \delta_2]$ is denoted by $\Gamma(x^0)$. Then

$$2\pi \cdot \frac{\partial U(x)}{\partial n_x}\bigg|_{x=x^0} = \frac{\partial}{\partial n_x}\left(\int\limits_{\Gamma(x^0)} \left(\frac{\partial}{\partial n_y}\ln|x-y|\right)\Phi(y)\,dl_y \right)\Bigg|_{x=x^0} + \frac{\partial}{\partial n_x}\left(\int\limits_{\Gamma\setminus\Gamma(x^0)} \left(\frac{\partial}{\partial n_y}\ln|x-y|\right)\Phi(y)\,dl_y \right)\Bigg|_{x=x^0}$$
$$\equiv (V_0\Phi)(x^0) + (V_1\Phi)(x^0)$$

A point $(0,\tau)$ is an interior one of the domain Ω if $\tau > 0$ is small. Hence,

$$(V_0\Phi)(x^0) = \lim_{\tau\to 0}\frac{\partial}{\partial\tau}\int\limits_{\xi_0-\delta_1}^{\xi_0+\delta_2} \frac{\tau}{(t-\xi_0)^2+\tau^2}\,\varphi(t)\,dt$$

Let $[\alpha,\beta] \subset [\xi_0-\delta_1,\, \xi_0+\delta_2]$ and $\xi_0 \notin [\alpha,\beta]$. Then,

$$\lim_{\tau\to 0}\frac{\partial}{\partial\tau}\int\limits_{\alpha}^{\beta}\frac{\tau}{(t-\xi_0)^2+\tau^2}\,\varphi(t)\,dt = \lim_{\tau\to 0}\int\limits_{\alpha}^{\beta}\frac{(t-\xi_0)^2-\tau^2}{\left((t-\xi_0)^2+\tau^2\right)^2}\,\varphi(t)\,dt = \int\limits_{\alpha}^{\beta}\frac{\varphi(t)}{(t-\xi_0)^2}\,dt$$

Setting $\delta_0 = 0.5\cdot\min(\delta_1,\delta_2)$, we note that

$$\frac{\partial}{\partial\tau}\int\limits_{\xi_0-\delta_0}^{\xi_0+\delta_0}\frac{\tau}{(t-\xi_0)^2+\tau^2}\,\varphi(t)\,dt = -\int\limits_{\xi_0-\delta_0}^{\xi_0+\delta_0}\frac{\partial}{\partial t}\left(\frac{t-\xi_0}{(t-\xi_0)^2+\tau^2}\right)\varphi(t)\,dt$$

Following [3], after repeated integration by parts we obtain

$$\lim_{\tau\to 0}\frac{\partial}{\partial\tau}\int\limits_{\xi_0-\delta_0}^{\xi_0+\delta_0}\frac{\tau}{(t-\xi_0)^2+\tau^2}\,\varphi(t)\,dt = -\int\limits_{\xi_0-\delta_0}^{\xi_0+\delta_0}\ln|t-\xi_0|\,\varphi''(t)\,dt + \left(\varphi'(t)\ln|t-\xi_0| - \frac{\varphi(t)}{t-\xi_0}\right)\Bigg|_{\xi_0-\delta_0}^{\xi_0+\delta_0} \tag{10}$$

Denote $I_0 = [\xi_0-\delta_0,\, \xi_0+\delta_0]$. Since $\Gamma(x_0) = [\xi_0-\delta_1,\, \xi_0+\delta_2]$ in our coordinate system, we get

$$(V_0\Phi)(x^0) = \int\limits_{\Gamma(x^0)\setminus I_0}\frac{\varphi(t)}{(t-\xi_0)^2}\,dt - \int\limits_{I_0}\ln|t-\xi_0|\,\varphi''(t)\,dt + \left(\varphi'(t)\ln|t-\xi_0| - \frac{\varphi(t)}{t-\xi_0}\right)\Bigg|_{\xi_0-\delta_0}^{\xi_0+\delta_0}$$

In order to perform the further mathematical treatment, it is necessary to compute derivatives of the function φ approximately.

Let $\xi \in (s_j, s_{j+1/2}]$. Denote

$$K_m(\xi,t) = \left(\frac{\partial^m}{\partial s^m}K(s,t)\right)\bigg|_{s=\xi}, \qquad m = 1,\,2,\,\ldots$$

Theorem 4. *If $\xi \in (s_j, s_{j+1/2}]$, then the estimate*

$$\left| \frac{d^m}{ds^m}\varphi(s)\bigg|_{s=\xi} - f^{(m)}(\xi) + \sum_{t_j^{(n)}\notin[s_j,s_{j+1}]} A_j^{(n)}K_m\left(\xi, t_j^{(n)}\right)\left(\varphi(t_j^{(n)}) - \varphi(s_j)\right) \right| \leqslant \frac{\mathrm{const}}{(\xi-s_j)^m}\cdot e^{-c\sqrt{n}}$$

holds for the weights and nodes of quadrature formula (5).

Proof. Since

$$K(s,t) \equiv 0$$

for $s,\, t \in [s_j, s_{j+1}]$, it follows from equation (3) that for each $s \in [s_j, s_{j+1}]$

$$\varphi(s) = -\int\limits_{[0,T]\setminus[s_j,s_{j+1}]} K(s,t)\left(\varphi(t)-\varphi(s_j)\right)dt + f(s) - \varphi(s_j)$$

Differentiating this equality m times with respect to s for $s = \xi$, we obtain

$$\frac{d^m}{ds^m}\,\varphi(s)\bigg|_{s=\xi} = -\int\limits_{[0,T]\setminus[s_j,s_{j+1}]} K_m(s,t)\,\big(\varphi(t)-\varphi(s_j)\big)\,dt + f^{(m)}(\xi)$$

In order to evaluate the integral in the right-hand side of the above equality, we use the composite quadrature formula (5) with the nodes that do not belong to the segment $[s_j, s_{j+1}]$.

Now we consider the segment $[s_{j-1/2}, s_j]$. The analysis of derivatives of the function $K(s,t)$ performed in [1] demonstrates that for any integer $k > 0$ the integrand can be continued analytically from the segment

$$\big[s_j - (s_j - s_{j-1/2})(1+\Theta_j)^{-k+1},\ s_j - (s_j - s_{j-1/2})(1+\Theta_j)^{-k}\big]$$

of the real axis into the circle of the complex plane with its center at the midpoint of this segment and with its radius

$$r_k = 0.5\,(s_j - s_{j-1/2})\big(1-(1+\Theta_j)^{-1}\big)(1+\Theta_j)^{2-k}$$

The analytic function obtained is bounded within this circle by

$$\frac{\text{const}}{(\xi - s_j)^{m+1}}\,(1+\Theta_j)^{-k\lambda_j}$$

Generalizing the technique proposed in [1] for estimation of the quadrature error, we complete our proof of Theorem 4.

Suppose a point ξ_0 belongs to the segment $[t_k, t_{k-1}]$ that lies in $[s_j, s_{j+1/2}]$. In (10) we replace the functions $\varphi''(t)$, $\varphi'(t)$, and $\varphi(t)$ by the interpolating Chebyshev polynomials with $n_{j,k}$ nodes on each of the segments $[\xi_0-\delta_1,\, \xi_0-\delta_0]$, $[\xi_0-\delta_0,\, \xi_0+\delta_0]$, and $[\xi_0+\delta_0,\, \xi_0+\delta_2]$, using the approximate values of $\varphi_n(t)$ and the values of derivatives that were computed by the method proposed in the proof of Theorem 2. Then, the integrals in the right-hand side of (10) can be evaluated analytically. In order to compute $(V_1\Phi)(x^0)$, we shall use the quadrature formula (5) with nodes that do not belong to the segment $[\xi_0-\delta_1, \xi_0+\delta_2]$.

As a result, we constructed a method that allows us to compute a value of the normal derivative for the solution to problem (1) at any fixed boundary point x^0 with an error of order $C(x_0)\,e^{-c\sqrt{n}}$, where $C(x^0)$ is a constant that depends only on a position of the point x^0 relative to the corner points of the boundary.

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