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Multigrid methods with skew-Hermitian based smoothers for the convection–diffusion problem with dominant convection

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Abstract: The convection–diffusion equation with dominant convection is considered on a uniform grid of central difference scheme. The multigrid method is used for solving the strongly nonsymmetric systems of linear algebraic equations with positive definite coefficient matrices. Two-step skew-Hermitian iterative methods are utilized for the first time as a smoothing procedure. It is demonstrated that using the proper smoothers enables to improve the convergence of the multigrid method. The robustness of the smoothers with respect to variation of the Peclet number is shown by local Fourier analysis and numerical experiments.

Keywords: convection–diffusion equation, multigrid methods, smoothing procedure, product-type skew-Hermitian triangular splitting, local Fourier analysis, convergence

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Многосеточные методы с косо-эрмитовыми сглаживателями для задач конвекции–диффузии с преобладающей конвекцией

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Аннотация: Уравнение конвекции–диффузии с преобладающей конвекцией рассматривается на равномерной сетке центрально-разностной схемы. Многосеточный метод используется для решения сильно несимметричных систем линейных алгебраических уравнений с положительно определенными матрицами коэффициентов. Двухшаговые косоэрмитовы итерационные методы впервые используются в качестве сглаживающей процедуры. Демонстрируется, что надлежащий выбор сглаживателей позволяет улучшить сходимость многосеточного метода. Локальный фурье-анализ и численные эксперименты приводят к выводу об устойчивости сглаживателей по отношению к изменению числа Пекле.

Ключевые слова: уравнение конвекции–диффузии, многосеточные методы, сглаживающая процедура, модифицированное эрмитово и косоэрмитово расщепление матрицы, локальный фурье-анализ, сходимость

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1. Introduction. The need to solve the convection–diffusion equation arises in the mathematical modeling of a great number of real processes. Especially many problems appear when the highest derivative has a small parameter. Various approaches for solving this problem are proposed. They are related to both different approximation methods of convective terms, which significantly affect the properties of the resulting non-symmetric matrix, and the construction of special grids [1–4]. Moreover, this class of problems is a test one when studying the convergence of iteration methods for solving non-self-adjoint systems of linear equations. Applying upwind differences for approximation of the first order derivatives, we can obtain a M-matrix [5], whereas using of the central differences allows to get a positive definite matrix [6].

We have used central difference approximation of convective terms. In this case, the resulting system of linear algebraic equations (SLAE) is a strongly non-symmetric one [7]. A new class of product-type skew-Hermitian triangular splitting iteration methods (PSTS) has been created to solve this type of linear systems [8, 9]. In addition, it was shown that these methods are effective as preconditioners for Krylov subspace methods. Any method in this class has the same behavior as the Gauss-Seidel iteration, i.e. it quickly reduces the high-frequency components of error frequencies but not the low-frequency ones [10]. This is the necessary property of the smoother of a multigrid method (MGM) [11–14], which is a successful tool for the solution of the SLAEs associated with discretization of boundary-value problems.

One of the universal methods for solving difference equations is the Fedorenko geometric multigrid method (GMG) [11] based on the use of a sequence of nested grids and transition operators from one grid to another. In this method, the solution process starts with the coarsest mesh. The resulting solution is interpolated to a fine mesh and used as an initial approximation in some iterative process, which requires a relatively small number of iterations to achieve a given accuracy.

The GMG for the elliptic problems with anisotropic discontinuous coefficients has been investigated in [15]. The authors of this paper have considered a 3-D diffusion problem with general boundary conditions and studied two iterative smoothers: the Chebyshev operator polynomial and a rational function. A geometric multigrid solver for the compact discontinuous Galerkin method was considered in [16] by constructing a hierarchy of coarser meshes using an agglomeration method that handles arbitrary element shapes and sizes. This method extends to other discontinuous Galerkin discretizations. In [17] authors compare the performance of seven different element agglomeration algorithms on unstructured triangular/tetrahedral meshes used as part of a geometric multigrid.

Nevertheless, there are classes of problems for which geometric techniques are too difficult to apply or cannot be used at all. They can be solved by the algebraic multigrid methods (AMG), introduced in [18–20]. In AMG the construction of auxiliary SLAE is carried out from the original SLAE by algebraic methods, i.e. without using information about the computational grid. This is especially actual when solving complex problems on unstructured grids. A comparative analysis of the classical algebraic and geometric approaches in the MGM was performed in [21]. The author of this work presented universal multigrid technology (UMT), which is a kind of geometric multigrid methods, developed for the numerical solution of boundary value problems. The UMT is a computing technology for use in software products arranged according to the “black box” principle.

We have used the AMG with the PMIS coarsening algorithm [22–23] for solving the incompressible unsteady Navier-Stokes equations, where the Hermitian/skew-Hermitian splitting (HSS) and the skew-Hermitian triangular splitting (STS) methods have been used as smoothers [24]. Convergence analysis of the MGM with the HSS based smoothers for the second-order non-self-adjoint elliptic problems has been done in [25]. In [26] authors have investigated the performance of smoothers relying on the HSS and of the augmented Lagrangian splittings applied to MAC (Marker-and-Cell) discretization of the Oseen problem. Local Fourier analysis (LFA) has been implemented in [12, 14] for 2-D lid-driven cavity problem and both steady and unsteady flows have been considered there. The LFA of the block-structured multigrid relaxation schemes for the staggered finite-difference discretization (MAC scheme) of the Stokes equations has been performed in [27]. A parallel implementation of the AMG for solving a system of linear equations generated by a finite-volume discretization of the Navier-Stokes equations on unstructured grids has been considered in [28].

The multigrid efficiency depends on the adjustment of its components to the problem being solved. We improve the convergence of the MGM by choosing special smoothers. Multigrid convergence with the PSTS smoothers is proved. A smoothing method may be called robust if it works for all small values of the Peclet number Pe for the convection–diffusion problems. Local Fourier analysis and numerical experiments indicate that the PSTS-multigrid method (multigrid method with the PSTS based smoothers) is quite efficient. In this regard, the use of the multigrid method with the proposed skew-Hermitian based smoothers will be advantageous in solving SLAEs with a strongly nonsymmetric matrix of coefficients, which arise in the process of studying problems of hydrodynamics in particular.

2. Convection–diffusion problem. Let us consider the steady convection–diffusion problem with the Dirichlet boundary condition

$$\begin{cases} -\dfrac{1}{Pe}\,\triangle\, u + \dfrac{1}{2}\left[v_1\dfrac{\partial u}{\partial x} + v_2\dfrac{\partial u}{\partial y} + \dfrac{\partial(v_1 u)}{\partial x} + \dfrac{\partial(v_2 u)}{\partial y}\right] = F, \quad \text{in} \quad \Omega, \\ u = 0, \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Pe is the Peclet number, $\Omega = (0,1)\times(0,1)$, $\partial\Omega$ is the boundary of the domain Ω, $v_i = v_i(x,y)(i = 1,2)$ and $F = F(x,y)$ are such continuous functions that the exact solution of the problem (1) has the following form

$$u(x,y) = \exp(xy)\sin(\pi x)\sin(\pi y).$$

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Moreover,

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \tag{2}$$

that follows from the medium incompressibility condition of the convection–diffusion equation (1).

When the central difference scheme on a uniform grid

$$\Omega_h = \{(x, y) \in \Omega \mid x = ih,\ y = jh,\ i, j = 1, 2, \dots, N\},$$

with $N \times N$ interior nodes is applied to the discretization of the convection–diffusion equation (1), we obtain a system of linear algebraic equations with the linear operator A_h: $E(\Omega_h) \to E(\Omega_h)$, where $E(\Omega_h)$ is the linear space of grid functions defined on Ω_h.

The matrix A, corresponding to the operator A_h, is a sparse strongly non-symmetric and positive definite one. As a consequence, we arrive to the following SLAE

$$Au = b, \quad u, b \in \mathbb{R}^n, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $n = N^2$ and the coefficient matrix A is [9, 29]

$$A = I \otimes P + Q \otimes I + R,$$

where symbol $\otimes$ stands for the Kronecker product;

$$P = \operatorname{tridiag}(-1, 4, -1) \in \mathbb{R}^{N\times N} \quad \text{and} \quad Q = \operatorname{tridiag}(-1, 0, -1) \in \mathbb{R}^{N\times N}$$

are N-by-N tridiagonal matrices;

$$R = \operatorname{tridiag}(-P_j, B_j, P_j) \in \mathbb{R}^{n\times n}$$

is an n-by-n block tridiagonal matrix;

$$P_j = \operatorname{diag}([P]_{j,k}) \in \mathbb{R}^{N\times N}, \quad j, k = 1, 2, \dots, N-1$$

are N-by-N diagonal matrices;

$$B_j = \operatorname{tridiag}(-[B]_{j,k}, 0, [B]_{j,k}) \in \mathbb{R}^{N\times N}, \quad j, k = 1, 2, \dots, N$$

are N-by-N tridiagonal matrices; in its turn, the elements $[P]_{j,k}$ and $[B]_{j,k}$ read:

$$\begin{cases} [P]_{j,k} = \dfrac{Pe\cdot h}{4}(v_{2(k,j)} + v_{2(k,j+1)}), & k = 1, 2, \dots, N, \qquad j = 1, 2, \dots, N-1, \\ [B]_{j,k} = \dfrac{Pe\cdot h}{4}(v_{1(k,j)} + v_{1(k+1,j)}), & k = 1, 2, \dots, N-1, \quad j = 1, 2, \dots, N, \end{cases}$$

with

$$v_{i(k,j)} = v_i(x_k, y_j), \quad i = 1, 2; \quad k, j = 1, 2, \dots, N.$$

Naturally, the matrix A can be split into its symmetric and skew-symmetric parts [5, 30] as

$$A = A_0 + A_1, \tag{4}$$

where

$$A_0 = \frac{1}{2}\left(A + A^T\right), \quad A_1 = \frac{1}{2}\left(A - A^T\right), \tag{5}$$

and A^T is the transposed matrix of A. Positive definiteness of the matrix A means that for all $x \in \mathbb{R}^n \setminus \{0\}$, $x^T A x > 0$. This fact leads to the conclusion that the symmetric matrix A_0 is also positive definite, and $\operatorname{diag}(A_1) = 0$. Let in some matrix norm $|||\cdot|||$ the inequality $|||A_0||| \ll |||A_1|||$ holds. Then the matrix A is called strongly non-symmetric one.

In this case, following the work [7], we split the skew-symmetric matrix A_1 into two parts

$$A_1 = K_L + K_U, \tag{6}$$

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where K_L and K_U are the strictly lower and the strictly upper triangular matrices respectively. Obviously that $K_L = -K_U^T$.

Based on the relations (4)–(6) the authors of [8, 9] have presented the product-type skew-Hermitian triangular splitting (PSTS) iteration methods for solving (3). Their main point is as follows:

The PSTS iteration methods [8, 9]: An initial value $u^{(0)}$ and two positive acceleration ω and τ are given. Let $u^{(k)}$ is a sequence of iterative approximations. Then for $k = 0, 1, 2, \dots$ until $u^{(k)}$ will converge, we compute

$$u^{(k+1)} = G(\omega, \tau)u^{(k)} + \tau B(\omega)^{-1}b, \tag{7}$$

where

$$G(\omega, \tau) = B(\omega)^{-1}\left(B(\omega) - \tau A\right)$$

with matrix $B(\omega)$, which is equal to

$$B(\omega) = \left(B_c + \frac{\omega}{2}\widehat{K}_L\right) B_c^{-1} \left(B_c + \frac{\omega}{2}\widehat{K}_U\right). \tag{8}$$

Here $\widehat{K}_L = K_L + F_0$, $\widehat{K}_U = K_U - F_0$, $F_0 \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $B_c \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix. Obviously, that $\widehat{K}_L = -\widehat{K}_U^T$, $A_1 = (K_L + F_0) + (K_U - F_0) = \widehat{K}_L + \widehat{K}_U$.

Let $B_0(\omega)$ and $B_1(\omega)$ be the symmetric and the skew-symmetric parts of the matrix $B(\omega)$ respectively:

$$B(\omega) = B_0(\omega) + B_1(\omega), \tag{9}$$

where

$$\begin{cases} B_0(\omega) = \dfrac{1}{2}\left(B(\omega) + B(\omega)^T\right) = B_c + \left(\dfrac{\omega}{2}\right)^2 \widehat{K}_L B_c^{-1} \widehat{K}_U, \\[2mm] B_1(\omega) = \dfrac{1}{2}\left(B(\omega) - B(\omega)^T\right) = \dfrac{\omega}{2}\left(\widehat{K}_L + \widehat{K}_U\right) = \dfrac{\omega}{2} A_1. \end{cases}$$

Suppose that

$$0 < \alpha_h I \leqslant A_0 \leqslant \beta_h I, \quad 0 < \alpha_c I \leqslant B_c \leqslant \beta_c I, \tag{10}$$

$$\alpha_l I \leqslant \widehat{K}_L B_c^{-1} \widehat{K}_U \leqslant \beta_l I, \quad \alpha_s I \leqslant B_0(\omega) \leqslant \beta_s I, \tag{11}$$

where I is an identity matrix. Since A_0 and B_c are symmetric positive definite matrices, $B_0(\omega)$ and $\widehat{K}_L B_c^{-1} \widehat{K}_U$ are symmetric matrices, and A_1 is a skew-symmetric matrix, the bounds α_ψ and β_ψ, $\psi = h, c, l, s$ can be easily expressed through the smallest and the largest eigenvalues or singular values of the corresponding matrix [8, 9].

We require positive definiteness of the matrix $B(\omega)$ from (9). Since the matrix $\widehat{K}_L B_c^{-1} \widehat{K}_U$ is negative definite, then $B_0(\omega) > 0$, if the parameter $\omega \in (0, \omega_{\max})$, where $\omega_{\max} = 2\sqrt{\left(-\dfrac{\alpha_c}{\alpha_l}\right)}$.

Notice, that α_l and β_l satisfy the following inequalities [9]:

$$\alpha_s \geqslant \alpha_c + \left(\frac{\omega}{2}\right)^2 \alpha_l, \qquad \beta_s \leqslant \beta_c + \left(\frac{\omega}{2}\right)^2 \beta_l. \tag{12}$$

Theorem 1 [9]. Let the matrices A and $B(\omega)$ be positive definite. If parameters ω and τ satisfy the inequalities

$$0 < \tau < \omega, \qquad 0 < \omega < \omega_{\max},$$

and

$$0 < \tau < \frac{2}{\Theta}, \qquad \Theta = \frac{\beta_h}{\alpha_c + \left(\dfrac{\omega}{2}\right)^2 \alpha_l}, \tag{13}$$

then the PSTS iteration method will be convergent, i.e. the spectral radius $\rho(G(\omega, \tau))$ of its iteration matrix $G(\omega, \tau)$ is less than 1.

When $F_0 = 0$, the PSTS method reduces to the *skew-symmetric product triangular splitting* (SPTS) iteration method studied in [31]. It is a generalization of the *modified skew-Hermitian triangular splitting* (MSTS) iteration method established in [29].

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Here we need two special versions of the SPTS iteration method (SPTS(1) and SPTS(2), see [31, 32]). We set $B_c = I$, $F_0 = 0$, $\omega = 2\tau$ for the SPTS(1) and $F_0 = 0$, $\omega = 2$ and the same B_c as that adopted in [32] for the SPTS(2), namely

$$B_c = \text{diag}\,(d_1, d_2, \dots, d_n),$$

with

$$d_i = \frac{1}{2}\sum_{j=1}^{n} \left|[A_0 + K_U - K_L]_{ij}\right|$$

being the i-th row-sum of the matrix $(A_0 + K_U - K_L)$.

We consider the PSTS method when $B_c = I$, $\omega = 2\tau$ and $(F_0 + K_L)$ is an unitary matrix [8, 9] and investigate the MGM with three skew-symmetric smoothers for solving non-symmetric system of linear equations (3). The application of the multigrid method, which is robust for the test problem (1), to the governing equations of computational fluid dynamics can enable to solve them effectively [14].

3. Convergence of the PSTS-multigrid method. There are several different ways to prove the convergence of the MGM, depending on the assumptions we are making. Usually, the smoothing and approximating properties are used to prove the convergence [13, 33]. Based on them, we are establishing convergence of the PSTS-multigrid method.

For this aim, we use some notations and theoretical results from [34, 35].

Let $H_1 \subset H_2 \subset \dots$ be a family of nested finite-dimensional linear spaces. The dimension of H_m is n_m and the inner product is denoted by $(.,.)_m$ with $\|.\|_m$ being the corresponding norm in H_m, $m = 1, 2, \dots$.

We are interested in solving problem (3) in H_m. Let

$$A_m = \widetilde{A}_m + \widehat{A}_m,$$

where $\widetilde{A}_m$ is a symmetric positive definite operator in H_m. Let Q_m is the other symmetric positive definite operator in H_m and let the following condition for the spectral radius of the operator $G_m = Q_m^{-1}\widetilde{A}_m$ is satisfied:

$$\rho\,(G_m) = 1.$$

Using the operators $\widetilde{A}_m$ and Q_m we determine the energy norm

$$\|u\|_{\widetilde{A}_m} = (\widetilde{A}_m u, u)^{1/2}, \quad \forall u \in H_m,$$

and the inner of the elements from space H_m

$$(u, v)_{Q_m} = (Q_m u, v), \quad \forall u, v \in H_m.$$

Then we may define the following norms on H_m:

$$\begin{aligned} \|u\|_{s,m} &= (G_m^s u, u)^{1/2}, \quad s \text{ — real} \\ \|u\|_{1,m} &= (\widetilde{A}_m u, u)^{1/2} = \|u\|_m, \\ \|u\|_{0,m} &= (Q_m u, u)^{1/2}. \end{aligned}$$

Let us define the subspace $F_m \subset H_m$, $F_m = \{u \in H_m\colon (A_m u, v) = 0,\ \forall v \in H_{m-1}\}$.

Now we make three basic assumptions [34, 35]:

Assumption 1. There exists $v \in H_{m-1}$, γ, $0 < \gamma \leqslant 1$ and $\delta < \infty$ such that for $\forall\ u \in H_m$

$$\|u - v\|^2_{s,m} \leqslant \delta^\gamma \|u\|^2_{1+\gamma,m}$$

Assumption 2. There exists η_m, $\eta_m \to 0$ $(m \to \infty)$ such that

$$\left|\left(\widehat{A}_m u, v\right)_m\right| < \eta_m\, \|u\|_{1,m}\, \|v\|_{1,m}$$

for $\forall\ u \in F_m$, $\forall v \in H_m$, m is a positive integer.

Assumption 3. There exists $\mu_m, \mu_m \to 0 \quad (m \to \infty)$ such that

$$\left|\left(\widehat{A}_m u, v\right)_m\right| \leqslant \mu_m\, \|u\|_{1,m}\, \|v\|_{0,m}$$

for $\forall u, v \in H_m$, m is a positive integer.

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It is supposed that η_m, μ_m are small enough.

In [35] it was shown that Assumptions 1–3 were satisfied for sufficiently wide class of elliptic boundary problems in two dimensional bounded domains with different boundary conditions.

Let us consider two-grid algorithm. Denote the exact solution of the problem (3) by u^*. Let u^0 is an initial guess, u^1 is a problem solution after smoothing and u^2 is a problem solution after MGM-iteration. The aim is to estimate the energy-norm of the contraction number:

$$\sigma = \sup \frac{\|u^2 - u^*\|_1}{\|u^0 - u^*\|_1}, \quad u^0 \neq u^*.$$

Denote the error by $e^i = u^i - u^*$, $i = 0, 1, 2$.

Theorem 2 [35]. Let the three basic assumptions for the two-grid method be satisfied. Furthermore, let the following smoothing assumption also be valid: there exist Δ $(0 < \Delta < \infty)$ and $\vartheta > 0$ such that

$$\left\|e^1\right\|_1^2 + \vartheta \left\|e^1\right\|_2^2 \leqslant (1 + \mu\Delta) \left\|e^0\right\|_1^2 \tag{14}$$

with $\mu{=}\mu_k$ from Assumption 3. Then $\sigma \leqslant \widetilde{\sigma}$ for the two-grid contraction number, where

$$\widetilde{\sigma} \equiv \widetilde{\sigma}(\epsilon) \equiv \sup \left\{ \left[\frac{\xi^2 + \epsilon^2 \varsigma^2 + 2\epsilon\eta\xi\varsigma}{1 + \delta^{-1}\vartheta \left(\dfrac{1-\eta}{1+\eta}\right)^{2/\gamma} \xi^{2/\gamma}} (1 + \mu\Delta) \right]^{1/2} \; : \; \xi^2 + \zeta^2 - 2\eta\xi\zeta \leqslant 1, \;\; \zeta, \xi \geqslant 0 \right\},$$

$$\xi = \frac{\|e^1 + u^*\|_1}{\|e^1\|_1}, \qquad \varsigma = \frac{\|u^*\|_1}{\|e^1\|_1},$$

where constants ϑ and Δ depend on properties of smoothing system and constants $\gamma, \delta, \eta, \mu$ taken from assumptions 1, 2, 3 respectively, while ϵ is calculation accuracy.

In [35] the proof of the same theorem for the MGM is given. For further considerations we need one more theorem from [34].

Theorem 3 [34]. Let the problem (3) be solved by iterative method (7)–(8) written in the form of

$$u^{(n+1)} = u^{(n)} - \widetilde{B}^{-1}\left(Au^{(n)} - b\right), \qquad \widetilde{A} = A_0,$$

and let the three basic assumptions be satisfied. If there exists the constant $\vartheta > 0$ so that the following inequality

$$\widetilde{B} + \widetilde{B}^* - \widetilde{A} \geqslant \vartheta \left(\widetilde{A} - \widetilde{B}\right)^* Q^{-1} \left(\widetilde{A} - \widetilde{B}\right), \tag{15}$$

is performed, then the smoothing assumption (14) will be satisfied with

$$\Delta = (1 + \vartheta)\, \beta\, (\mu\beta + 2), \tag{16}$$

where

$$\beta = \left[\rho\left[\widetilde{B}^{-1}\widetilde{A}\left(\widetilde{B}^{-1}\right)^*\widetilde{A}\right]\right]^{1/2}.$$

Now we can prove the convergence of the method.

Theorem 4. For the method (7), (8) there exists the constant $\vartheta > 0$ such that inequality (15) holds.

Proof. In the case of employing method (7), (8) $\widetilde{B} = 1/\tau B$, $Q = A_0$. Consider inequality (15) and transform its left and right parts. Using (9) we obtain

$$\widetilde{B} + \widetilde{B}^* - \widetilde{A} = 2/\tau B_0 - A_0, \tag{17}$$

$$\left(\widetilde{A} - \widetilde{B}\right)^* \widetilde{A}^{-1} \left(\widetilde{A} - \widetilde{B}\right) = \left(A_0 - \frac{1}{\tau}B^*\right) A_0^{-1} \left(A_0 - \frac{1}{\tau}B\right) = \\ = A_0 - \frac{1}{\tau}B^* - \frac{1}{\tau}B + \frac{1}{\tau^2}B^* A_0^{-1} B = A_0 - \frac{2}{\tau}B_0 + \frac{1}{\tau^2}B^* A_0^{-1} B. \tag{18}$$

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Introduce a new matrix S in accordance with the definition given below:

$$S = \frac{2}{\tau}B_0 - A_0. \tag{19}$$

Then from (13) we obtain

$$\frac{\beta_h}{\alpha_c + \left(\frac{\omega}{2}\right)^2 \alpha_l} < \frac{2}{\tau}.$$

It follows from (10)–(13) that the value β_h is the upper bound of the self-adjoint positive definite matrix A_0. On the other hand, the value in the denominator of the last fraction is positive (this results from the requirement that the operator B_0 is positive definite or from the requirement that the operator B is dissipative (which is equivalent)) and does not exceed the lower estimate of the $B_0(\omega)$ bound (equal to α_s). Therefore, for any value of the parameter $0 < \omega < \omega_{\max}$, for two self-adjoint positive-definite operators, the inequality $B_0 > \tau/2A_0 > 0$ will be satisfied.

Note that the last criterion is a sufficient condition for the convergence of the family of iterative methods based on the symmetric/skew-symmetric (Hermitian/skew-Hermitian) splitting of the coefficient matrix considered previously in [6, 7, 31, 32].

Thus, for acceleration parameters ω, τ satisfying the conditions of the Theorem 1, it is valid

$$S = S^* > 0.$$

Utilizing the relations (17)–(19) we obtain that the inequality (15) is rewritten in the following form:

$$S \geqslant \vartheta\left(-S + \frac{1}{\tau^2}B^*A_0^{-1}B\right).$$

For any Hermitian positively (non-negatively) definite matrix S there exists a unique Hermitian positively (non-negatively) definite matrix Q such that $Q^2 = S$. The matrix Q is called the (arithmetic) square root of the matrix S and is denoted $S^{1/2}$ [5].

Multiplying the left and right parts of this inequality on $S^{-1/2}$, we find straightforwardly that

$$I \geqslant \vartheta\left(-I + \frac{1}{\tau^2}S^{-1/2}B^*A_0^{-1}BS^{-1/2}\right). \tag{20}$$

Denote $L = \frac{1}{\tau}A_0^{-1/2}BS^{-1/2}$. Then (20) is transformed to

$$I \geqslant \vartheta\left(L^*L - I\right).$$

If we set

$$\vartheta = \frac{1}{\|L^*L - I\|}, \tag{21}$$

then inequality (15) will be valid. The theorem is proved.

Thus, if the conditions of the Theorem 4 are fulfilled, then the two-grid method will converge for the PSTS smoother from (8), the parameter ϑ from (21) and Δ taken from (16).

Since the SPTS iteration methods are the special case of the PSTS ones, the convergence results of the PSTS-multigrid are applicable to them as well. Notice, that analogous results can be obtained for the MSTS-multigrid.

4. Local Fourier analysis of the PSTS-multigrid method. The one-grid local Fourier analysis (LFA) (or the smoothing analysis) and the two-grid LFA are the main tools for quantitative estimates of the MGM convergence [12–14]. Within LFA the basic discrete operators are considered to be formally extended to infinite grid and boundary conditions are not taken into account. A direct application of the LFA is not possible while dealing with operator A_h that is characterized by variable coefficients. Instead, the analysis is applied to the locally frozen operator at a fixed grid point [12].

Let us consider regular infinite fine grid G_h

$$G_h = \{x = (x, y),\ x = ih,\ y = jh,\ i, j \in \mathbb{Z}\}$$

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and coarse grid G_{2h}, which is obtained from G_h by the standard coarsening (i.e. by doubling the mesh size in both x and y directions)

$$G_{2h} = \{x = (x, y),\ x = 2ih,\ y = 2jh,\ i, j \in \mathbb{Z}\}.$$

We perform LFA of the PSTS-, SPTS(1)- and SPTS(2)-multigrid methods and numerically obtain the smoothing factors and the two-grid convergence factors [13]. Denote these factors as

$$\mu_{\rm loc}(S_h), \quad \rho_{\rm loc}(M_h^{2h}),$$

where

$$M_h^{2h} = S_h^{\nu_2} K_h^{2h} S_h^{\nu_1}$$

is the transition operator of the two-grid method, ν_1 and ν_2 are the numbers of pre- and post-smoothing steps,

$$K_h^{2h} = I_h - P_{2h}^h A_{2h}^{-1} R_h^{2h} A_h$$

is the coarse-grid correction operator, S_h is the smoothing operator, P_{2h}^h is the prolongation operator and R_h^{2h} is the restriction operator.

We use LFA as a general tool for realistic estimation of the smoothing properties of the relaxation methods and the convergence properties of two-grid methods. We assume that the operators A_h, A_{2h}, R_h^{2h} and P_{2h}^h are represented by patterns for G_h and G_{2h} [13].

In order to measure the smoothing properties of pattern relaxation methods, the real coarse-grid correction is replaced by an ideal coarse-grid correction operator, which annihilates the low-frequency error components and leaves the high-frequency components unchanged. The results of the Fourier smoothing analysis for the MGM with the skew-symmetric smoothers and the large Peclet numbers are listed in Table 1. We take the coefficients of the convective terms as $v_1 = 1, v_2 = 1$ and consider the SPTS(k) ($k = 1, 2$) and the PSTS iteration methods as the smoothers. From Table 1 we see that for all the skew-symmetric methods, the coefficient factor $\mu_{\rm loc}$ is less than unity (except for SPTS(1) and SPTS(2), when $Pe = 10^7$). Therefore the SPTS(k) ($k = 1, 2$) and the PSTS are effective as the smoothers for the MGM when solving the problem (1). Moreover, the last method has the best smoothing properties. However, the smoothing properties of all tested methods deteriorate with increasing of the Peclet number.

Table 1. Smoothing factors $\mu_{\rm loc}$

Pe	SPTS(1)	SPTS(2)	PSTS
10^3	0.7459	0.6881	0.2961
10^4	0.8583	0.8150	0.3776
10^5	0.9399	0.8961	0.6989
10^6	0.9998	0.9912	0.8961
10^7	1.0065	1.0011	0.9903

The rate of asymptotic convergence of the two-grid method $\rho_{\rm loc}$ for various number m of smoothing iterations ($m = 1, 3, 5$) and at the same velocity values is shown in Table 2. The two-grid Fourier analysis allows to define the optimal number of smoothing iterations for the large Peclet numbers. Our investigations demonstrate that further increasing the number of smoothing iterations ($m > 5$) does not lead to a significant decrease in the value of ρ_{loc}. For the SPTS(k) ($k = 1, 2$) $\rho_{\rm loc} > 1$ when the one smoothing iteration is used only. Accordingly to the results of Table 2, we conclude that the PSTS-multigrid methods are robust for solving the convection–diffusion problem (1) with dominant convection.

Table 2. Asymptotic convergence factors $\rho_{\rm loc}$ at m smoothing iterations

Pe	m	SPTS(1)	SPTS(2)	PSTS
10^3	1	> 1	> 1	0.5949
	3	0.8895	0.7231	0.3866
	5	0.8514	0.6815	0.3191
10^4	1	> 1	> 1	0.7254
	3	0.9898	0.9531	0.5433
	5	0.9718	0.9462	0.5319
10^5	1	> 1	> 1	0.8582
	3	0.9989	0.9934	0.6769
	5	0.9951	0.9823	0.6741
10^6	1	> 1	> 1	0.9932
	3	> 1	0.9991	0.8985
	5	0.9967	0.9932	0.8935
10^7	1	> 1	> 1	> 1
	3	> 1	0.9999	0.9129
	5	0.9993	0.9968	0.9081

5. Numerical results. Numerical experiments were carried out using the MGM with the SPTS(1)-, SPTS(2)-, PSTS- and Gauss-Seidel (GS)- based smoothers for solving convection–diffusion problem (1). The four velocity coefficients $v = (v_1, v_2)^T$, given in Table 3, automatically satisfy the constraint (2) of the medium incompressibility.

Table 3. The test problems

Problem	$v_1(x, y)$	$v_2(x, y)$
1	1	-1
2	$1 - 2x$	$2y - 1$
3	$x + y$	$x - y$
4	$\sin(2\pi x)$	$-2\pi y \cdot \cos(2\pi x)$

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Let us briefly describe the organization of calculations using the MGM [12–13]. For this goal we consider a system of grid equations:

$$A_h u^h = b^h.$$

The interpolation operator P from a coarse grid H to a fine grid h allows the operator A_H to be represented on a coarse grid in the form

$$A_H = R A_h P,$$

where $R = P^T$. Then solution correction step reads

$$u^h_{\text{new}} = u^h_{\text{old}} + P \cdot e^H.$$

The correction e^H is the exact solution of the equation

$$A_H e^H = r^H,$$

where $r^H = R r^h$ and $r^h = b^h - A_h u^h_{\text{old}}$. In this case, before and after correction of the solution, μ_1 pre-smoothing and μ_2 post-smoothing steps are made utilizing some iteration methods.

Thus, the MGM using the solution correction scheme is the following sequence of steps:

1. We make μ_1 approximations of the solution on the grid h using the iterative method PSTS (GS, SPTS(1) or SPTS(2))(pre-smoothing).
2. The residual $r^h = b^h - A_h u^h_{\text{old}} \in V_h$ is projected into space V_H, i.e., $r^H = R r^h$.

Table 4. Iteration count and CPU time (in seconds) for the MGM with different smoothers, $h = 1/512$, $H = 1/4$

Smoother	GS	SPTS(1)	SPTS(2)	PSTS
Pe	IT (CPU)	IT (CPU)	IT (CPU)	IT (CPU)
Problem 1				
10^3	58 (5.84)	13 (0.36)	9 (0.21)	4 (0.34)
10^4	–	18 (1.45)	15 (1.36)	5 (1.43)
10^5	–	–	26 (6.91)	5 (6.67)
10^6	–	–	31 (11.75)	7 (10.64)
10^7	–	–	–	10 (15.02)
Problem 2				
10^3	66 (6.53)	13 (0.37)	10 (0.22)	5 (0.36)
10^4	–	22 (1.78)	14 (1.34)	6 (1.57)
10^5	–	–	–	7 (7.25)
10^6	–	–	–	9 (9.39)
10^7	–	–	–	14 (19.77)
Problem 3				
10^3	71 (7.57)	15 (0.39)	10 (0.23)	4 (0.34)
10^4	–	24 (1.92)	16 (1.37)	6 (1.58)
10^5	–	–	28 (7.04)	6 (6.99)
10^6	–	–	39 (13.17)	9 (10.06)
10^7	–	–	–	12 (17.56)
Problem 4				
10^3	–	22 (0.54)	15 (0.34)	6 (0.47)
10^4	–	–	21 (2.42)	7 (2.64)
10^5	–	–	–	7 (17.58)
10^6	–	–	–	15 (23.16)
10^7	–	–	–	19 (35.82)

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Table 5. Iteration count and CPU time (in seconds) for the MGM with different smoothers, $h = 1/512$, $H = 1/32$

Smoother	GS	SPTS(1)	SPTS(2)	PSTS
Pe	IT (CPU)	IT (CPU)	IT (CPU)	IT (CPU)
Problem 1				
10^3	56 (5.78)	12 (0.34)	8 (0.19)	4 (0.31)
10^4	–	17 (1.41)	14 (1.31)	4 (1.38)
10^5	–	33 (8.81)	24 (6.88)	4 (6.57)
10^6	–	–	27 (9.12)	6 (8.96)
10^7	–	–	33 (14.77)	7 (10.54)
Problem 2				
10^3	64 (6.39)	12 (0.34)	9 (0.20)	4 (0.32)
10^4	–	21 (1.68)	12 (1.30)	5 (1.44)
10^5	–	39 (9.18)	28 (7.19)	6 (6.84)
10^6	–	–	39 (10.56)	7 (9.13)
10^7	–	–	41 (15.98)	9 (12.77)
Problem 3				
10^3	70 (7.49)	14 (0.36)	9 (0.21)	4 (0.32)
10^4	–	23 (1.88)	14 (1.32)	5 (1.48)
10^5	–	38 (8.29)	27 (6.99)	5 (6.75)
10^6	–	–	36 (10.02)	7 (9.12)
10^7	–	–	40 (16.66)	10 (14.11)
Problem 4				
10^3	91 (10.57)	21 (0.52)	13 (0.29)	5 (0.44)
10^4	–	28 (4.12)	20 (2.39)	6 (2.55)
10^5	–	–	–	6 (16.49)
10^6	–	–	–	11 (15.65)
10^7	–	–	–	16 (26.78)

3. An approximate solution $A_H e^H = r^H$ is found on a coarse grid. For this aim, a γ of cycles of the multigrid method is realized recursively.
4. The correction e^H is interpolated to the fine grid and the solution is refined: $u^h_{\text{new}} = u^h_{\text{old}} + P \cdot e^H$.
5. μ_2 approximations of the solution are made on a fine mesh to suppress the interpolation error (post-smoothing).

Depending on the number γ of recursive calls, different types of cycles are distinguished. When $\gamma = 1$, the V-cycle takes place, and when $\gamma = 2$, the W-cycle occurs [13].

Our calculations are carried out using a V-cycle with 5 pre-smoothing and zero post-smoothing steps. All components of the MGM, apart from the smoothers, are standard ones. The prolongation operator is bilinear interpolation and the restriction operator is the full weighting operator [13]. For the PSTS a smoothing step is one full PSTS iteration. It should be noted that for the PSTS smoothing we have to solve a linear system with the coefficient matrix $B(\omega)$ defined in (8). This SLAE is solved inexactly by the inner GMRES(10)[30] with the tolerance 10^{-6}. No preconditioner is used for the inner iterations. Since $B(\omega)$ is a product of two triangular matrices in SPTS(1) and SPTS(2), this system of linear equations can be solved easily. For all tested methods we take the parameter τ to be the experimentally optimal one.

It is known that for non-self-adjoint problems the mesh size of the coarsest grid have to be sufficiently small [33] to ensure the multigrid convergence. To compare dependence of the MGM convergence with the considered smoothers on the coarsest grid, we choose the finest grid mesh size as $h = 1/512$ $\left(h = \dfrac{1}{N+1}\right)$ and the coarsest grid mesh size as $H = 1/4$ or $1/32$ respectively.

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Numerical experiments are performed for $Pe = 10^3 \div 10^7$ and the corresponding results are listed in Tables 4, 5. All iterations are started from zero vector and terminated when

$$\frac{\|r^{(p)}\|_2}{\|r^{(0)}\|_2} \leqslant 10^{-6}.$$

Here $r^{(p)} = b - Au^{(p)}$ is the residual vector of the SLAE (3) at the current iterate $u^{(p)}$ and $r^{(0)}$ is the initial residual. Our comparisons are done for the number of iteration steps (denoted by "IT") and the elapsed CPU time denoted by "CPU". We show total time for solving the SLAE. The symbol ("–") indicates that method failed to converge.

All experiments are performed in MATLAB (version R2018b) with a machine precision 10^{-16} on a personal computer with 3.60 GHz central processing unit (Intel(R) Core(TM) i7-7700), 16.00 GB memory, and Windows 10 operating system. From the numerical experiments, we can see that the PSTS-multigrid method is required far less iteration steps than GS, SPTS(1)- and SPTS(2)-multigrid methods for all test problems. The skew-symmetry coefficient $Pe_h = Pe \cdot h/2$ has the great influence on the behaviour of the MGM with the SPTS(1) and SPTS(2) smoothers. As this coefficient increases, the convergence rate of the multigrid method decreases. But the PSTS-multigrid does not have such a strong dependence on the skew-symmetry coefficient, which is a positive quality of the method.

The SPTS(2)-multigrid outperforms the PSTS-multigrid in computational time due to the presence inner iterations in the last method, but for $Pe = 10^5 \div 10^7$, CPU time becomes less for the PSTS-MGM. Moreover, the SPTS(2)-multigrid is ahead of the SPTS(1)- and GS-multigrid methods with respect to both number of iteration steps and CPU time for all tested problems.

Among the considered smoothers, only PSTS-multigrid successfully solves four problems for all Peclet number values and $H = 1/4, 1/32$. The GS-multigrid doesn't converge at Pe more than 10^4 for all the tested problems and for $Pe = 10^3$, when $H = 1/4$ for the problem 4. In contrast, at $Pe = 10^4 \div 10^7$, the PSTS-multigrid method shows fast convergence speed, and the number of its iteration steps do not change significantly when Pe becomes large.

It follows from numerical results that PSTS-multigrid has smaller dependence on the mesh size of the coarsest grid than GS-, SPTS(1)- and SPTS(2)-multigrid. When $H = 1/4$, SPTS(1)-multigrid is not convergent at Pe more than 10^5 for all test problems and for Pe more than 10^4 for the Problem 4 with a highly variable velocity field. SPTS(2)-multigrid doesn't converge for some problems when Pe more than 10^5, whereas PSTS-multigrid is always convergent. All considered methods tend to decrease the number of iterations and CPU time as the coarse grid mesh size decreases.

6. Conclusions. Numerical experiments carried out for the model steady-state convection–diffusion problem with dominant convection have shown efficiency of the PSTS-multigrid and its advantage over the GS and SPTS(k)($k = 1, 2$) based smoothers. The application of the PSTS smoothers for the MGM allows solving this problem for small values of Pe. This modification of the smoother enables to use multigrid method without restriction on the coefficients of equation and mesh size of the coarsest grid for the solution of arising SLAE. It doesn't require the diagonal dominance of corresponding matrix. Numerical experiments show that it is possible to use efficient point relaxation methods in connection with standard coarsening for the convection–diffusion problem with dominant convection.

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