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Energy Conservation Law for Yee-Type Schemes

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Abstract: It is shown that in a class of the finite-difference schemes for Maxwell equations with improved dispersion properties an analog of energy conservation law holds. The existence of this law is useful in applying the schemes and provides their numerical stability. It is assumed that the equations describe a medium consisting of different materials with frequency dispersion.

Keywords: finite-difference scheme, Maxwell Equations, Yee scheme, finite-difference analog of energy conservation law, Maxwell equations with frequency dispersion, contact interface.

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Закон сохранения энергии для аналогов схем Йи

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Аннотация: Показано, что в классе конечно-разностных схем для уравнений Максвелла с улучшенными дисперсионными свойствами выполняется аналог закона сохранения энергии. Существование этого закона полезно при использовании схем и обеспечивает их численную устойчивость. Подразумевается, что уравнения описывают среду, состоящую из различных материалов с частотной дисперсией.

Ключевые слова: конечно-разностная схема, уравнения Максвелла, схема Йи, разностный аналог закона сохранения энергии, уравнения Максвелла с частотной дисперсией, контактная граница.

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1. Introduction. The existence of analogs of conservation laws of initial differential equations is a positive property of finite-difference schemes. For example, in modeling the impact of laser pulse on the material the key value is the pulse energy absorbed by the material. This value determines the modification of the material [1–6] . The reproducing of energy conservation law by a scheme is very important in this case. Especially if the modeling demands large computational resources and it is impossible to find a solution with high accuracy in respect to all features. Even two-dimensional laser-matter interaction problems [2, 5, 6] are resource-intensive, not to mention three-dimensional ones [3, 4, 7]. In addition, the existence of the energy conservation law provides numerical stability of the scheme [8].

In the presented paper it is shown that a class of the schemes [9] for Maxwell equations taking into account optic properties of materials has energy conservation law similar to the law for the initial equations. These schemes are based on the Yee scheme [10], but they are more advanced compared to it in respect to the dispersion properties and a description of an interface between different materials.

The paper is organized as follows. In section 2, the mathematical model and the finite-difference scheme are described. Then, in section 3 the proof and a view of the energy conservation law are presented. In Conclusion some notes about possible generalization of the received results are discussed.

2. Mathematical model. The equations of the Drude model [2, 6] are used as initial equations in the presented paper. The model is described using the following physical quantities: refractive index n, the coefficient of laser radiation absorption α (due to, for example, photoionization), current $\boldsymbol{J}$ of free electrons

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with a density ρ, mass m and charge e. It is assumed that a friction with coefficient ν forces on free electrons. For brevity, we write dimensionless equations directly:

$$\frac{\partial\left(n^2\boldsymbol{E}\right)}{\partial t} = \omega_{\mathrm{p}}^2\boldsymbol{J} + c\,\mathrm{rot}\,\boldsymbol{B} - \alpha\boldsymbol{E}, \tag{1}$$

$$\frac{\partial\boldsymbol{J}}{\partial t} = -\boldsymbol{E} - \nu\boldsymbol{J}, \tag{2}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\,\mathrm{rot}\,\boldsymbol{E}, \tag{3}$$

$$\mathrm{div}\,\boldsymbol{B} = 0.$$

Here $\omega_{\mathrm{p}}^2 = 4\pi\rho e^2/m$ is a square of plasma frequency. To solve (1)–(3) a finite difference scheme using a cubic grid is employed. Each cubic cell of the grid with the edge size h and the center in the point $x_{i+1/2}, y_{j+1/2}, z_{k+1/2}$ (or $i+1/2, j+1/2, k+1/2$) is occupied by a material with optic coefficients $n^2_{i+1/2,j+1/2,k+1/2}$, $\omega^2_{\mathrm{p}\,i+1/2,j+1/2,k+1/2}$, $\alpha_{i+1/2,j+1/2,k+1/2}$, $\nu_{i+1/2,j+1/2,k+1/2}$. The perpendicular to the cube face components of electric field and the current are calculated in the center of the face. Different cells can contain various materials. Therefore, the component of electric field can be discontinuous. For this reason, two values for electric field and current with an upper index $(\pm)$, calculated by optical coefficients in neighbor cells, are used. Magnetic field components parallel to the cube edge are calculated in the center of the edge. Such the scheme is similar to the Yee scheme [9]. The difference is that the values with tilde (see below) are used for the approximations of operator rot. This provides better reproducing of dispersion properties of (1)–(3) and a more reliable solution on the interface. The finite-difference equations (1)–(2) have the following view [9]:

$$n^2_{i\pm1/2,j+1/2,k+1/2}\frac{E_{x\,\cdots}^{(\pm)n+1/2} - E_{x\,\cdots}^{(\pm)n-1/2}}{\tau} = \omega^2_{\mathrm{p}\,i\pm1/2,j+1/2,k+1/2}\frac{J_{x\,\cdots}^{(\pm)n+1/2} + J_{x\,\cdots}^{(\pm)n-1/2}}{2} + \\ + \frac{\tilde B^n_{z\,i,j+1,k+1/2} - \tilde B^n_{z\,i,j,k+1/2}}{hc^{-1}} - \frac{\tilde B^n_{y\,i,j+1/2,k+1} - \tilde B^n_{y\,i,j+1/2,k}}{hc^{-1}} - \alpha_{i\pm1/2,j+1/2,k+1/2}\frac{E_{x\,\cdots}^{(\pm)n+1/2} + E_{x\,\cdots}^{(\pm)n-1/2}}{2}, \tag{4}$$

$$\frac{J_{x\,\cdots}^{(\pm)n+1/2} - J_{x\,\cdots}^{(\pm)n-1/2}}{\tau} = -\frac{E_{x\,\cdots}^{(\pm)n+1/2} + E_{x\,\cdots}^{(\pm)n-1/2}}{2} - \nu_{i\pm1/2,j+1/2,k+1/2}\frac{J_{x\,\cdots}^{(\pm)n+1/2} + J_{x\,\cdots}^{(\pm)n-1/2}}{2}. \tag{5}$$

For brevity, in (4), (5) the notation i, $j+1/2$, $k+1/2$ is replaced by "$\cdots$".

$$n^2_{i+1/2,j\pm1/2,k+1/2}\frac{E_{y\,\cdots}^{(\pm)n+1/2} - E_{y\,\cdots}^{(\pm)n-1/2}}{\tau} = \omega^2_{\mathrm{p}\,i+1/2,j\pm1/2,k+1/2}\frac{J_{y\,\cdots}^{(\pm)n+1/2} + J_{y\,\cdots}^{(\pm)n-1/2}}{2} + \\ + \frac{\tilde B^n_{x\,i+1/2,j,k+1} - \tilde B^n_{x\,i+1/2,j,k}}{hc^{-1}} - \frac{\tilde B^n_{z\,i+1,j,k+1/2} - \tilde B^n_{z\,i,j,k+1/2}}{hc^{-1}} - \alpha_{i+1/2,j\pm1/2,k+1/2}\frac{E_{y\,\cdots}^{(\pm)n+1/2} + E_{y\,\cdots}^{(\pm)n-1/2}}{2}, \tag{6}$$

$$\frac{J_{y\,\cdots}^{(\pm)n+1/2} - J_{y\,\cdots}^{(\pm)n-1/2}}{\tau} = -\frac{E_{y\,\cdots}^{(\pm)n+1/2} + E_{y\,\cdots}^{(\pm)n-1/2}}{2} - \nu_{i+1/2,j\pm1/2,k+1/2}\frac{J_{y\,\cdots}^{(\pm)n+1/2} + J_{y\,\cdots}^{(\pm)n-1/2}}{2}. \tag{7}$$

For brevity, in (6), (7) the notation $i+1/2$, j, $k+1/2$ is replaced by "$\cdots$".

$$n^2_{i+1/2,j+1/2,k\pm1/2}\frac{E_{z\,\cdots}^{(\pm)n+1/2} - E_{z\,\cdots}^{(\pm)n-1/2}}{\tau} = \omega^2_{\mathrm{p}\,i+1/2,j+1/2,k\pm1/2}\frac{J_{z\,\cdots}^{(\pm)n+1/2} + J_{z\,\cdots}^{(\pm)n-1/2}}{2} + \\ + \frac{\tilde B^n_{y\,i+1,j+1/2,k} - \tilde B^n_{y\,i,j+1/2,k}}{hc^{-1}} - \frac{\tilde B^n_{x\,i+1/2,j+1,k} - \tilde B^n_{x\,i+1/2,j,k}}{hc^{-1}} - \alpha_{i+1/2,j+1/2,k\pm1/2}\frac{E_{z\,\cdots}^{(\pm)n+1/2} + E_{z\,\cdots}^{(\pm)n-1/2}}{2}, \tag{8}$$

$$\frac{J_{z\,\cdots}^{(\pm)n+1/2} - J_{z\,\cdots}^{(\pm)n-1/2}}{\tau} = -\frac{E_{z\,\cdots}^{(\pm)n+1/2} + E_{z\,\cdots}^{(\pm)n-1/2}}{2} - \nu_{i+1/2,j+1/2,k\pm1/2}\frac{J_{z\,\cdots}^{(\pm)n+1/2} + J_{z\,\cdots}^{(\pm)n-1/2}}{2}. \tag{9}$$

The symbol "$\cdots$" means $i+1/2$, $j+1/2$, k in (8), (9).

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In its turn, the components of magnetic field with tilde are defined as

$$\tilde{B}^n_{x\,i+1/2,j,k} = B^n_{x\,i+1/2,j,k} + \sigma(B^n_{x\,i+3/2,j,k} - 2B^n_{x\,i+1/2,j,k} + B^n_{x\,i-1/2,j,k}), \quad (10)$$

$$\tilde{B}^n_{y\,i,j+1/2,k} = B^n_{y\,i,j+1/2,k} + \sigma(B^n_{y\,i,j+3/2,k} - 2B^n_{y\,i,j+1/2,k} + B^n_{y\,i,j-1/2,k}), \quad (11)$$

$$\tilde{B}^n_{z\,i,j,k+1/2} = B^n_{z\,i,j,k+1/2} + \sigma(B^n_{z\,i,j,k+3/2} - 2B^n_{z\,i,j,k+1/2} + B^n_{z\,i,j,k-1/2}). \quad (12)$$

The equation (3) is approximated in the following way:

$$\frac{B^{n+1}_{x\,i+1/2,j,k} - B^n_{x\,i+1/2,j,k}}{c\tau} = \frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2} - \tilde{E}^{n+1/2}_{y\,i+1/2,j,k-1/2}}{h} - \frac{\tilde{E}^{n+1/2}_{z\,i+1/2,j+1/2,k} - \tilde{E}^{n+1/2}_{z\,i+1/2,j-1/2,k}}{h}, \quad (13)$$

$$\frac{B^{n+1}_{y\,i,j+1/2,k} - B^n_{y\,i,j+1/2,k}}{c\tau} = \frac{\tilde{E}^{n+1/2}_{z\,i+1/2,j+1/2,k} - \tilde{E}^{n+1/2}_{z\,i-1/2,j+1/2,k}}{h} - \frac{\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2} - \tilde{E}^{n+1/2}_{x\,i,j+1/2,k-1/2}}{h}, \quad (14)$$

$$\frac{B^{n+1}_{z\,i,j,k+1/2} - B^n_{z\,i,j,k+1/2}}{c\tau} = \frac{\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2} - \tilde{E}^{n+1/2}_{x\,i,j-1/2,k+1/2}}{h} - \frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2} - \tilde{E}^{n+1/2}_{y\,i-1/2,j,k+1/2}}{h}, \quad (15)$$

where

$$\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2} = \frac{\tilde{E}^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} + \tilde{E}^{(-)n+1/2}_{x\,i,j+1/2,k+1/2}}{2}, \quad (16)$$

$$\tilde{E}^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} = E^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} + 2\sigma(E^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2} - E^{(+)n+1/2}_{x\,i,j+1/2,k+1/2}), \quad (17)$$

$$\tilde{E}^{(-)n+1/2}_{x\,i,j+1/2,k+1/2} = E^{(-)n+1/2}_{x\,i,j+1/2,k+1/2} + 2\sigma(E^{(+)n+1/2}_{x\,i-1,j+1/2,k+1/2} - E^{(-)n+1/2}_{x\,i,j+1/2,k+1/2}), \quad (18)$$

$$\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2} = \frac{\tilde{E}^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} + \tilde{E}^{(-)n+1/2}_{y\,i+1/2,j,k+1/2}}{2}, \quad (19)$$

$$\tilde{E}^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} = E^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} + 2\sigma(E^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2} - E^{(+)n+1/2}_{y\,i+1/2,j,k+1/2}), \quad (20)$$

$$\tilde{E}^{(-)n+1/2}_{y\,i+1/2,j,k+1/2} = E^{(-)n+1/2}_{y\,i+1/2,j,k+1/2} + 2\sigma(E^{(+)n+1/2}_{y\,i+1/2,j-1,k+1/2} - E^{(-)n+1/2}_{y\,i+1/2,j,k+1/2}), \quad (21)$$

$$\tilde{E}^{n+1/2}_{z\,i+1/2,j+1/2,k} = \frac{\tilde{E}^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} + \tilde{E}^{(-)n+1/2}_{z\,i+1/2,j+1/2,k}}{2}, \quad (22)$$

$$\tilde{E}^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} = E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} + 2\sigma(E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1} - E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k}), \quad (23)$$

$$\tilde{E}^{(-)n+1/2}_{z\,i+1/2,j+1/2,k} = E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k} + 2\sigma(E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k-1} - E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k}). \quad (24)$$

Similarly to (16)–(24) notations with tilde for current J can be introduced. The value $\sigma = 1/12$ is necessary for better reproducing of dispersion properties of initial equations [9].

3. The derivation of energy conservation law. To derive the energy conservation law for (4)–(24) an infinite computational domain and a solution tending to zero at infinity are considered. This allows not to pay attention to the limits of the summation indexes. Omitting for brevity subindexes x, $j+1/2$, $k+1/2$ in (4)–(5) and writing down them separately for $(+)$ and $(-)$, we get

$$n^2_{i+1/2}\frac{E^{(+)n+1/2}_i - E^{(+)n-1/2}_i}{\tau} - \omega^2_{\mathrm{p}\,i+1/2}\frac{J^{(+)n+1/2}_i + J^{(+)n-1/2}_i}{2} + \alpha_{i+1/2}\frac{E^{(+)n+1/2}_i + E^{(+)n-1/2}_i}{2} = \\ = \frac{\tilde{B}^n_{z\,i,j+1,k+1/2} - \tilde{B}^n_{z\,i,j,k+1/2}}{hc^{-1}} - \frac{\tilde{B}^n_{y\,i,j+1/2,k+1} - \tilde{B}^n_{y\,i,j+1/2,k}}{hc^{-1}}, \quad (25)$$

$$\frac{J^{(+)n+1/2}_i - J^{(+)n-1/2}_i}{\tau} = -\frac{E^{(+)n+1/2}_i + E^{(+)n-1/2}_i}{2} - \nu_{i+1/2}\frac{J^{(+)n+1/2}_i + J^{(+)n-1/2}_i}{2}, \quad (26)$$

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$$n^2_{i-1/2}\frac{E_i^{(-)n+1/2}-E_i^{(-)n-1/2}}{\tau}-\omega^2_{\mathrm{p}\,i-1/2}\frac{J_i^{(-)n+1/2}+J_i^{(-)n-1/2}}{2}+\alpha_{i-1/2}\frac{E_i^{(-)n+1/2}+E_i^{(-)n-1/2}}{2}=$$

$$=\frac{\tilde{B}^n_{z\,i,j+1,k+1/2}-\tilde{B}^n_{z\,i,j,k+1/2}}{hc^{-1}}-\frac{\tilde{B}^n_{y\,i,j+1/2,k+1}-\tilde{B}^n_{y\,i,j+1/2,k}}{hc^{-1}}, \tag{27}$$

$$\frac{J_i^{(-)n+1/2}-J_i^{(-)n-1/2}}{\tau}=-\frac{E_i^{(-)n+1/2}+E_i^{(-)n-1/2}}{2}-\nu_{i-1/2}\frac{J_i^{(-)n+1/2}+J_i^{(-)n-1/2}}{2}. \tag{28}$$

Multiplying (25) by $\left(\tilde{E}_i^{(+)n+1/2}+\tilde{E}_i^{(+)n-1/2}\right)$, (26) by $\omega^2_{\mathrm{p}\,i+1/2}\left(\tilde{J}_i^{(+)n+1/2}+\tilde{J}_i^{(+)n-1/2}\right)$, (27) by $\left(\tilde{E}_i^{(-)n+1/2}+\tilde{E}_i^{(-)n-1/2}\right)$, (28) by $\omega^2_{\mathrm{p}\,i-1/2}\left(\tilde{J}_i^{(-)n+1/2}+\tilde{J}_i^{(-)n-1/2}\right)$, and summing the results of the multiplication with each other (here notations with tilde for current J are similar to (16)–(24)) and over all nodes of the grid, we obtain

$$\sum_{j,k}\sum_i\Bigg\{\frac{n^2_{i+1/2}}{\tau}\left(\tilde{E}_i^{(+)n+1/2}+\tilde{E}_i^{(+)n-1/2}\right)\left(E_i^{(+)n+1/2}-E_i^{(+)n-1/2}\right)+$$
$$+\frac{\omega^2_{\mathrm{p}\,i+1/2}}{\tau}\left(\tilde{J}_i^{(+)n+1/2}+\tilde{J}_i^{(+)n-1/2}\right)\left(J_i^{(+)n+1/2}-J_i^{(+)n-1/2}\right)+$$
$$+\frac{\omega^2_{\mathrm{p}\,i+1/2}}{2}\Big(\left(\tilde{J}_i^{(+)n+1/2}+\tilde{J}_i^{(+)n-1/2}\right)\left(E_i^{(+)n+1/2}+E_i^{(+)n-1/2}\right)-$$
$$-\left(\tilde{E}_i^{(+)n+1/2}+\tilde{E}_i^{(+)n-1/2}\right)\left(J_i^{(+)n+1/2}+J_i^{(+)n-1/2}\right)\Big)+$$
$$+\frac{\alpha_{i+1/2}}{2}\left(\tilde{E}_i^{(+)n+1/2}+\tilde{E}_i^{(+)n-1/2}\right)\left(E_i^{(+)n+1/2}+E_i^{(+)n-1/2}\right)+$$
$$+\frac{\nu_{i+1/2}\,\omega^2_{\mathrm{p}\,i+1/2}}{2}\left(\tilde{J}_i^{(+)n+1/2}+\tilde{J}_i^{(+)n-1/2}\right)\left(J_i^{(+)n+1/2}+J_i^{(+)n-1/2}\right)\Bigg\}+$$
$$+\sum_{j,k}\sum_i\Bigg\{\frac{n^2_{i-1/2}}{\tau}\left(\tilde{E}_i^{(-)n+1/2}+\tilde{E}_i^{(-)n-1/2}\right)\left(E_i^{(-)n+1/2}-E_i^{(-)n-1/2}\right)+$$
$$+\frac{\omega^2_{\mathrm{p}\,i-1/2}}{\tau}\left(\tilde{J}_i^{(-)n+1/2}+\tilde{J}_i^{(-)n-1/2}\right)\left(J_i^{(-)n+1/2}-J_i^{(-)n-1/2}\right)+$$
$$+\frac{\omega^2_{\mathrm{p}\,i-1/2}}{2}\Big(\left(\tilde{J}_i^{(-)n+1/2}+\tilde{J}_i^{(-)n-1/2}\right)\left(E_i^{(-)n+1/2}+E_i^{(-)n-1/2}\right)-$$
$$-\left(\tilde{E}_i^{(-)n+1/2}+\tilde{E}_i^{(-)n-1/2}\right)\left(J_i^{(-)n+1/2}+J_i^{(-)n-1/2}\right)\Big)+$$
$$+\frac{\alpha_{i-1/2}}{2}\left(\tilde{E}_i^{(-)n+1/2}+\tilde{E}_i^{(-)n-1/2}\right)\left(E_i^{(-)n+1/2}+E_i^{(-)n-1/2}\right)+$$
$$+\frac{\nu_{i-1/2}\,\omega^2_{\mathrm{p}\,i-1/2}}{2}\left(\tilde{J}_i^{(-)n+1/2}+\tilde{J}_i^{(-)n-1/2}\right)\left(J_i^{(-)n+1/2}+J_i^{(-)n-1/2}\right)\Bigg\}=$$
$$=\sum_{j,k}\sum_i\left(\tilde{E}_i^{(+)n+1/2}+\tilde{E}_i^{(+)n-1/2}+\tilde{E}_i^{(-)n+1/2}+\tilde{E}_i^{(-)n-1/2}\right)\times$$
$$\times c\left(\frac{\tilde{B}^n_{z\,i,j+1,k+1/2}-\tilde{B}^n_{z\,i,j,k+1/2}}{h}-\frac{\tilde{B}^n_{y\,i,j+1/2,k+1}-\tilde{B}^n_{y\,i,j+1/2,k}}{h}\right).$$

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Let shift index i in the second sum to calculate optic coefficients in the first and second sum in the same points. This leads to the following continuation of the previous equality:

$$\sum_{j,k}\sum_{i}\Bigg\{\frac{n^2_{i+1/2}}{\tau}\Big[\Big(\tilde E_i^{(+)n+1/2}+\tilde E_i^{(+)n-1/2}\Big)\Big(E_i^{(+)n+1/2}-E_i^{(+)n-1/2}\Big)+ \\ +\Big(\tilde E_{i+1}^{(-)n+1/2}+\tilde E_{i+1}^{(-)n-1/2}\Big)\Big(E_{i+1}^{(-)n+1/2}-E_{i+1}^{(-)n-1/2}\Big)\Big]+ \\ +\frac{\omega^2_{\mathrm{p}\,i+1/2}}{\tau}\Big[\Big(\tilde J_i^{(+)n+1/2}+\tilde J_i^{(+)n-1/2}\Big)\Big(J_i^{(+)n+1/2}-J_i^{(+)n-1/2}\Big)+ \\ +\Big(\tilde J_{i+1}^{(-)n+1/2}+\tilde J_{i+1}^{(-)n-1/2}\Big)\Big(J_{i+1}^{(-)n+1/2}-J_{i+1}^{(-)n-1/2}\Big)\Big]+ \\ +\frac{\omega^2_{\mathrm{p}\,i+1/2}}{2}\Big[\Big(\tilde J_i^{(+)n+1/2}+\tilde J_i^{(+)n-1/2}\Big)\Big(E_i^{(+)n+1/2}+E_i^{(+)n-1/2}\Big)- \\ -\Big(\tilde E_i^{(+)n+1/2}+\tilde E_i^{(+)n-1/2}\Big)\Big(J_i^{(+)n+1/2}+J_i^{(+)n-1/2}\Big)+ \\ +\Big(\tilde J_{i+1}^{(-)n+1/2}+\tilde J_{i+1}^{(-)n-1/2}\Big)\Big(E_{i+1}^{(-)n+1/2}+E_{i+1}^{(-)n-1/2}\Big)- \\ -\Big(\tilde E_{i+1}^{(-)n+1/2}+\tilde E_{i+1}^{(-)n-1/2}\Big)\Big(J_{i+1}^{(-)n+1/2}+J_{i+1}^{(-)n-1/2}\Big)\Big]+D^n_{x\,i+1/2}\Bigg\}= \\ =2c\sum_{j,k}\sum_{i}\Big(\tilde E_i^{n+1/2}+\tilde E_i^{n-1/2}\Big)\left(\frac{\tilde B^n_{z\,i,j+1,k+1/2}-\tilde B^n_{z\,i,j,k+1/2}}{h}-\frac{\tilde B^n_{y\,i,j+1/2,k+1}-\tilde B^n_{y\,i,j+1/2,k}}{h}\right). \qquad (29)$$

Dissipative term $D^n_{x\,i+1/2}$ has a view

$$D^n_{x\,i+1/2}=\frac{\alpha_{i+1/2}}{2}\Big[\Big(\tilde E_i^{(+)n+1/2}+\tilde E_i^{(+)n-1/2}\Big)\Big(E_i^{(+)n+1/2}+E_i^{(+)n-1/2}\Big)+ \\ +\Big(\tilde E_{i+1}^{(-)n+1/2}+\tilde E_{i+1}^{(-)n-1/2}\Big)\Big(E_{i+1}^{(-)n+1/2}+E_{i+1}^{(-)n-1/2}\Big)\Big]+ \\ +\frac{\nu_{i+1/2}\omega^2_{\mathrm{p}\,i+1/2}}{\tau}\Big[\Big(\tilde J_i^{(+)n+1/2}+\tilde J_i^{(+)n-1/2}\Big)\Big(J_i^{(+)n+1/2}+J_i^{(+)n-1/2}\Big)+ \\ +\Big(\tilde J_{i+1}^{(-)n+1/2}+\tilde J_{i+1}^{(-)n-1/2}\Big)\Big(J_{i+1}^{(-)n+1/2}+J_{i+1}^{(-)n-1/2}\Big)\Big].$$

Using (16)–(18) and analogous formulas for J, equation (29) can be rewritten in the form

$$\sum_{j,k}\sum_{i}\Bigg\{\frac{n^2_{i+1/2}}{\tau}\Big[\Big(\tilde E_i^{(+)n+1/2}E_i^{(+)n+1/2}+\tilde E_{i+1}^{(-)n+1/2}E_{i+1}^{(-)n+1/2}\Big)- \\ -\Big(\tilde E_i^{(+)n-1/2}E_i^{(+)n-1/2}+\tilde E_{i+1}^{(-)n-1/2}E_{i+1}^{(-)n-1/2}\Big)\Big]+ \\ +\frac{\omega^2_{\mathrm{p}\,i+1/2}}{\tau}\Big[\Big(\tilde J_i^{(+)n+1/2}J_i^{(+)n+1/2}+\tilde J_{i+1}^{(-)n+1/2}J_{i+1}^{(-)n+1/2}\Big)- \\ -\Big(\tilde J_i^{(+)n-1/2}J_i^{(+)n-1/2}+\tilde J_{i+1}^{(-)n-1/2}J_{i+1}^{(-)n-1/2}\Big)\Big]+D^n_{x\,i+1/2}\Bigg\}= \\ =2c\sum_{j,k}\sum_{i}\Big(\tilde E_i^{n+1/2}+\tilde E_i^{n-1/2}\Big)\left(\frac{\tilde B^n_{z\,i,j+1,k+1/2}-\tilde B^n_{z\,i,j,k+1/2}}{h}-\frac{\tilde B^n_{y\,i,j+1/2,k+1}-\tilde B^n_{y\,i,j+1/2,k}}{h}\right). \qquad (30)$$

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It should be noted that for the interesting case $\sigma \leqslant 1/4$, the quantities such as $\tilde{E}_i^{(+)} E_i^{(+)} + \tilde{E}_{i+1}^{(-)} E_{i+1}^{(-)} \geqslant 0$. Indeed, using (17), (18), one can get

$$\tilde{E}_i^{(+)} E_i^{(+)} + \tilde{E}_{i+1}^{(-)} E_{i+1}^{(-)} = \left(E_i^{(+)} + 2\sigma\left(E_{i+1}^{(-)} - E_i^{(+)}\right)\right) E_i^{(+)} + \left(E_{i+1}^{(-)} + 2\sigma\left(E_i^{(+)} - E_{i+1}^{(-)}\right)\right) E_{i+1}^{(-)} =$$
$$= \left(E_i^{(+)}\right)^2 + \left(E_{i+1}^{(-)}\right)^2 - 2\sigma\left(E_i^{(+)} - E_{i+1}^{(-)}\right)^2 =$$
$$= \left(\left(E_i^{(+)}\right)^2 + \left(E_{i+1}^{(-)}\right)^2\right)\left(1 - 2\sigma(\cos\varphi - \sin\varphi)^2\right) \geqslant \left(\left(E_i^{(+)}\right)^2 + \left(E_{i+1}^{(-)}\right)^2\right)(1 - 4\sigma),$$

where

$$\cos\varphi = \frac{E_i^{(+)}}{\sqrt{\left(E_i^{(+)}\right)^2 + \left(E_{i+1}^{(-)}\right)^2}}, \qquad \sin\varphi = \frac{E_{i+1}^{(-)}}{\sqrt{\left(E_i^{(+)}\right)^2 + \left(E_{i+1}^{(-)}\right)^2}}.$$

The terms with τ in (30) are differences of positive values evaluated at the time layers $n + 1/2$ and $n - 1/2$ respectively. The dissipative term $D^n_{x\,i+1/2}$ is also positive.

Analogously, for brevity, let us write B_k^n instead of $B^n_{z\,i,j,k+1/2}$ and introduce values

$$\tilde{B}_k^{(+)} = B_k + 2\sigma(B_{k+1} - B_k), \quad \tilde{B}_k^{(-)} = B_k + 2\sigma(B_{k-1} - B_k).$$

In these notations (12) takes a form $\tilde{B}_k = (\tilde{B}_k^{(+)} + \tilde{B}_k^{(-)})/2$. Multiplying (15) one time by $\left(\tilde{B}_k^{(+)n+1} + \tilde{B}_k^{(+)n}\right)$, another time by $\left(\tilde{B}_k^{(-)n+1} + \tilde{B}_k^{(-)n}\right)$, then summing the resulting products together and over grid points analogously to the case of E_x, one obtains

$$\sum_{i,j}\sum_k \frac{1}{c\tau}\left\{\left(\tilde{B}_k^{(+)n+1} + \tilde{B}_k^{(+)n}\right)\left(B_k^{n+1} - B_k^n\right) + \left(\tilde{B}_k^{(-)n+1} + \tilde{B}_k^{(-)n}\right)\left(B_k^{n+1} - B_k^n\right)\right\} =$$
$$= \sum_{i,j}\sum_k \left(\tilde{B}_k^{(+)n+1} + \tilde{B}_k^{(+)n} + \tilde{B}_k^{(-)n+1} + \tilde{B}_k^{(-)n}\right) \times$$
$$\times \left(\frac{\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2} - \tilde{E}^{n+1/2}_{x\,i,j-1/2,k+1/2}}{h} - \frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2} - \tilde{E}^{n+1/2}_{y\,i-1/2,j,k+1/2}}{h}\right).$$

The left hand side of this equality can be transformed as

$$\sum_{i,j}\sum_k \frac{1}{c\tau}\left\{\left(\tilde{B}_k^{(+)n+1} + \tilde{B}_k^{(+)n}\right)\left(B_k^{n+1} - B_k^n\right) + \left(\tilde{B}_{k+1}^{(-)n+1} + \tilde{B}_{k+1}^{(-)n}\right)\left(B_{k+1}^{n+1} - B_{k+1}^n\right)\right\} =$$
$$= \sum_{i,j}\sum_k \frac{1}{c\tau}\left\{\left(\tilde{B}_k^{(+)n+1} B_k^{n+1} + \tilde{B}_{k+1}^{(-)n+1} B_{k+1}^{n+1}\right) - \left(\tilde{B}_k^{(+)n} B_k^n + \tilde{B}_{k+1}^{(-)n} B_{k+1}^n\right)\right\} =$$
$$= \sum_{i,j}\sum_k \frac{2}{c\tau}\left(\tilde{B}_k^{n+1} B_k^{n+1} - \tilde{B}_k^n B_k^n\right).$$

Finally, we arrive to the formula

$$\sum_{i,j}\sum_k \frac{2}{c\tau}\left(\tilde{B}_k^{n+1} B_k^{n+1} - \tilde{B}_k^n B_k^n\right) = 2\sum_{i,j}\sum_k \left(\tilde{B}_k^{n+1} + \tilde{B}_k^n\right) \times$$
$$\times \left(\frac{\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2} - \tilde{E}^{n+1/2}_{x\,i,j-1/2,k+1/2}}{h} - \frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2} - \tilde{E}^{n+1/2}_{y\,i-1/2,j,k+1/2}}{h}\right). \quad (31)$$

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The sum of the right-hand sides of (30) and (31), multiplied by the speed of light c, has the following form:

$$2c\sum_{i,j}\sum_{k}\left\{\left(\tilde{B}^{n+1}_{z\,i,j,k+1/2}+\tilde{B}^{n}_{z\,i,j,k+1/2}\right)\left[\frac{\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}-\tilde{E}^{n+1/2}_{x\,i,j-1/2,k+1/2}}{h}-\frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2}-\tilde{E}^{n+1/2}_{y\,i-1/2,j,k+1/2}}{h}\right]+\right.$$
$$\left.+\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}+\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}\right)\left[\frac{\tilde{B}^{n}_{z\,i,j+1,k+1/2}-\tilde{B}^{n}_{z\,i,j,k+1/2}}{h}-\frac{\tilde{B}^{n}_{y\,i,j+1/2,k+1}-\tilde{B}^{n}_{y\,i,j+1/2,k}}{h}\right]\right\}. \quad (32)$$

Rewriting expression $\sum\limits_{i,j,k}\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}+\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}\right)\left(\tilde{B}^{n}_{z\,i,j+1,k+1/2}-\tilde{B}^{n}_{z\,i,j,k+1/2}\right)$ as

$$\sum_{i,j,k}\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}+\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}\right)\tilde{B}^{n}_{z\,i,j+1,k+1/2}-\sum_{i,j,k}\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}+\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}\right)\tilde{B}^{n}_{z\,i,j,k+1/2}$$

and changing summation index $j+1$ to j in the first sum, one can rewrite (32) as

$$2c\sum_{i,j}\sum_{k}\left\{\tilde{B}^{n+1}_{z\,i,j,k+1/2}\frac{\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}-\tilde{E}^{n+1/2}_{x\,i,j-1/2,k+1/2}}{h}-\tilde{B}^{n}_{z\,i,j,k+1/2}\frac{\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}-\tilde{E}^{n-1/2}_{x\,i,j-1/2,k+1/2}}{h}-\right.$$
$$-\left(\tilde{B}^{n+1}_{z\,i,j,k+1/2}+\tilde{B}^{n}_{z\,i,j,k+1/2}\right)\frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2}-\tilde{E}^{n+1/2}_{y\,i-1/2,j,k+1/2}}{h}-$$
$$\left.-\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}+\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}\right)\frac{\tilde{B}^{n}_{y\,i,j+1/2,k+1}-\tilde{B}^{n}_{y\,i,j+1/2,k}}{h}\right\}. \quad (33)$$

From equations (30), (31), (33), we have

$$\frac{1}{\tau}\sum_{j,k}\sum_{i}\left\{\left[n^2_{i+1/2}\left(\tilde{E}^{(+)n+1/2}_{i}E^{(+)n+1/2}_{i}+\tilde{E}^{(-)n+1/2}_{i+1}E^{(-)n+1/2}_{i+1}\right)+\right.\right.$$
$$+\,\omega^2_{\mathrm{p}\,i+1/2}\left(\tilde{J}^{(+)n+1/2}_{i}J^{(+)n+1/2}_{i}+\tilde{J}^{(-)n+1/2}_{i+1}J^{(-)n+1/2}_{i+1}\right)+$$
$$\left.+2\,\tilde{B}^{n+1}_{z\,i,j,k+1/2}\left(B^{n+1}_{k}-(c\tau/h)\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}-\tilde{E}^{n+1/2}_{x\,i,j-1/2,k+1/2}\right)\right)\right]-$$
$$-\left[n^2_{i+1/2}\left(\tilde{E}^{(+)n-1/2}_{i}E^{(+)n-1/2}_{i}+\tilde{E}^{(-)n-1/2}_{i+1}E^{(-)n-1/2}_{i+1}\right)+\right.$$
$$+\,\omega^2_{\mathrm{p}\,i+1/2}\left(\tilde{J}^{(+)n-1/2}_{i}J^{(+)n-1/2}_{i}+\tilde{J}^{(-)n-1/2}_{i+1}J^{(-)n-1/2}_{i+1}\right)+$$
$$\left.\left.+\,2\tilde{B}^{n}_{z\,i,j,k+1/2}\left(B^{n}_{k}-(c\tau/h)\left(\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}-\tilde{E}^{n-1/2}_{x\,i,j-1/2,k+1/2}\right)\right)\right]\right\}=$$
$$=-\sum_{i,j}\sum_{k}D^{n}_{x\,i+1/2}-2c\sum_{i,j}\sum_{k}\left\{\left(\tilde{B}^{n+1}_{z\,i,j,k+1/2}+\tilde{B}^{n}_{zi,j,k+1/2}\right)\frac{\tilde{E}^{n+1/2}_{y\,i+1/2,j,k+1/2}-\tilde{E}^{n+1/2}_{y\,i-1/2,j,k+1/2}}{h}+\right.$$
$$\left.+\left(\tilde{E}^{n+1/2}_{x\,i,j+1/2,k+1/2}+\tilde{E}^{n-1/2}_{x\,i,j+1/2,k+1/2}\right)\frac{\tilde{B}^{n}_{y\,i,j+1/2,k+1}-\tilde{B}^{n}_{y\,i,j+1/2,k}}{h}\right\}.$$

Taking other components of electromagnetic field into account and carrying out similar manipulations, we obtain the conservation law

$$\sum_{i,j,k}\frac{\mathcal{E}^{n+1}_{i+1/2,j+1/2,k+1/2}-\mathcal{E}^{n}_{i+1/2,j+1/2,k+1/2}}{\tau}=-\sum_{i,j,k}\left(D^{n}_{x\,i+\frac{1}{2}}+D^{n}_{y\,j+\frac{1}{2}}+D^{n}_{z\,k+\frac{1}{2}}\right).$$

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Here $D^n_{y\,j+1/2}$, $D^n_{z\,k+1/2}$ are dissipative terms connected with y, z components of electric field in full analogy with the case of $D^n_{x\,i+1/2}$. The energy density is given by the formula

$$\begin{gathered}
\mathcal{E}^{n+1}_{i+1/2,j+1/2,k+1/2} = n^2_{i+1/2,j+1/2,k+1/2}\left(\tilde{E}^{(+)n+1/2}_{x\,i,j+1/2,k+1/2}E^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} + \tilde{E}^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2}E^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2}+\right.\\
+\tilde{E}^{(+)n+1/2}_{y\,i+1/2,j,k+1/2}E^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} + \tilde{E}^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2}E^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2} + \tilde{E}^{(+)n+1/2}_{z\,i+1/2,j+1/2,k}E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k}+\\
\left.+\tilde{E}^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1}E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1}\right) + \omega^2_{\mathrm{p}\,i+1/2,j+1/2,k+1/2}\left(\tilde{J}^{(+)n+1/2}_{x\,i,j+1/2,k+1/2}J^{(+)n+1/2}_{x\,i,j+1/2,k+1/2}+\right.\\
+\tilde{J}^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2}J^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2} + \tilde{J}^{(+)n+1/2}_{y\,i+1/2,j,k+1/2}J^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} + \tilde{J}^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2}J^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2}+\\
\left.+\tilde{J}^{(+)n+1/2}_{z\,i+1/2,j+1/2,k}J^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} + \tilde{J}^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1}J^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1}\right)+\\
+2\left[\tilde{B}^{n+1}_{x\,i+1/2,j,k}\left(B^{n+1}_{x\,i+1/2,j,k} + c\tau\left(\mathrm{rot}\,\tilde{\boldsymbol{E}}^{n+1/2}\right)_{x\,i+1/2,j,k}\right)+\right.\\
+\tilde{B}^{n+1}_{y\,i,j+1/2,k}\left(B^{n+1}_{y\,i,j+1/2,k} + c\tau\left(\mathrm{rot}\,\tilde{\boldsymbol{E}}^{n+1/2}\right)_{y\,i,j+1/2,k}\right)+\\
\left.+\tilde{B}^{n+1}_{z\,i,j,k+1/2}\left(B^{n+1}_{z\,i,j,k+1/2} + c\tau\left(\mathrm{rot}\,\tilde{\boldsymbol{E}}^{n+1/2}\right)_{z\,i,j,k+1/2}\right)\right]. \qquad (34)
\end{gathered}$$

Here $\left(\mathrm{rot}\,\boldsymbol{E}^{n+1/2}\right)_{x,y,z}$ are finite-difference analogues of the x, y, z components of $\mathrm{rot}\,\boldsymbol{E}$, calculated in the corresponding points.

Formula (34) is a quadratic form of $\boldsymbol{E}^{n+1/2}$, $\boldsymbol{J}^{n+1/2}$, $\boldsymbol{B}^{n+1}$. The diagonal part of the form, as it was mentioned above, is positive. Non-diagonal terms, associated with $\mathrm{rot}\,\tilde{\boldsymbol{E}}^{n+1/2}$ are proportional to $c\tau/h$. Thus, if $c\tau/h$ is sufficiently small, the quadratic form (34) for the energy $\mathcal{E}^{n+1}_{i+1/2,j+1/2,k+1/2}$ is positively definite. This provides numerical stability of the scheme [8].

The term $\tilde{B}^{n+1}_{z\,i,j,k+1/2}\left(B^{n+1}_{z\,i,j,k+1/2} + c\tau\left(\mathrm{rot}\,\tilde{\boldsymbol{E}}^{n+1/2}\right)_{z\,i,j,k+1/2}\right)$ appearing in (34) can be written shorter as $\tilde{B}^{n+1}_{z\,i,j,k+1/2}\,B^{n}_{z\,i,j,k+1/2}$. Similar notational simplification can be also done for the terms with x, y components of magnetic field. But in this form the belonging of the electromagnetic field to the certain step of time in the energy expression (34) becomes not obvious.

It is better to rewrite sums like $\sum\limits_{i,j,k}\tilde{B}^{n+1}_{z\,i,j,k+1/2}B^{n+1}_{z\,i,j,k+1/2}$ in the form

$$\begin{gathered}
\sum_{i,j,k}\left(\tilde{B}^{n+1}_{z\,i,j,k+1/2}B^{n+1}_{z\,i,j,k+1/2} + \tilde{B}^{n+1}_{z\,i+1,j,k+1/2}B^{n+1}_{z\,i+1,j,k+1/2}+\right.\\
\left.+\tilde{B}^{n+1}_{z\,i,j+1,k+1/2}B^{n+1}_{z\,i,j+1,k+1/2} + \tilde{B}^{n+1}_{z\,i+1,j+1,k+1/2}B^{n+1}_{z\,i+1,j+1,k+1/2}\right)/4
\end{gathered}$$

for a symmetry. This was not done in (34) for the sake of brevity.

To check the obtained results, two-dimensional test problems were solved. In the first problem a square computational domain with size 50×50 microns in $(x,\,z)$ plane is considered. The periodic boundary conditions in both coordinates are used. A strip $0 < z_1 < z < z_2 < 50$ μm is occupied by plasma with ω^2_{p} about or less a few tenths fs^{-2} and with n^2 about a few units. Concrete parameters are varied. The dissipative coefficients are assumed to be zero. There is a vacuum in the regions $z > z_2$ and $z < z_1$. The electromagnetic field has y-component of the magnetic field and x- and z-components of the electric field. Note that E_z has a discontinuity on the vacuum-plasma interface. Initially the magnetic field has a Gaussian distribution and $E_x = E_z = 0$. The scheme parameter σ is set to 0, 1/12, 1/6. The calculations show that in this problem the energy changes with time in 11–12-th decimal place.

Another solved problem is the propagation of a focused femtosecond laser pulse incident at an angle of $\pi/4$ onto a plate of ionized glass. The problem has 2D slab (x-z) geometry with nonzero E_x, E_z, B_y polarization. The pulse is initiated by boundary condition on $z = 0$. The details are presented in [9]. Initially the energy is put into the calculation region. But there is an interval when the pulse is more or less isolated from the boundaries. The calculations show that on this interval the energy changes in 5-th digit.

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4. Conclusion. Let us turn to summarizing the results. The analytical proof of the energy conservation law for the scheme (4)–(24) is given in the presented paper. The solution of 2D test problems shows that the finite-difference analog of the energy is conserved up to rounding errors in the case of perfectly isolated radiation. When the influence of the boundary is not negligible, but the boundary is sufficiently far from the electromagnetic pulse, the energy is conserved with high accuracy as well.

Now we specify some remarks. Very often in the laser-matter interaction modeling the carrier frequency ω is separated out. In this case, the derivative $\dfrac{\partial}{\partial t}$ is replaced by $\dfrac{\partial}{\partial t} - i\omega$ and values $\boldsymbol{E}$, $\boldsymbol{J}$, $\boldsymbol{B}$ become complex [6, 9]. In the scheme (4)–(24) it is necessary to replace expressions like $(f^{n+1} - f^n)/\tau$ by $\big((f^{n+1} - f^n)/\tau - i\omega(f^{n+1} + f^n)/2\big)$ [6, 9]. The energy conservation law has a similar view in this case. All computations above will be the same. The only difference is that multiplication by the complex conjugation values is required. For example, $\left(\tilde{E}_i^{(+)n+1/2} + \tilde{E}_i^{(+)n-1/2}\right)\left(E_i^{(+)n+1/2} - E_i^{(+)n-1/2}\right)$ should be replaced by $\left(\left(\tilde{E}_i^{(+)n+1/2} + \tilde{E}_i^{t(+t)n-1/2}\right)^*\left(E_i^{(+)n+1/2} - E_i^{(+)n-1/2}\right) + \text{cc}\right)$.

The parameter σ is constant in the above considerations. But the energy conservation law also holds if it depends on the coordinates and on the component of the electromagnetic field. In this case, we have

$$
\begin{aligned}
\tilde{E}^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} &= E^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} + 2\sigma^{Ex}_{i+1/2,j+1/2,k+1/2}\left(E^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2} - E^{(+)n+1/2}_{x\,i,j+1/2,k+1/2}\right),\\
\tilde{E}^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2} &= E^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2} + 2\sigma^{Ex}_{i+1/2,j+1/2,k+1/2}\left(E^{(+)n+1/2}_{x\,i,j+1/2,k+1/2} - E^{(-)n+1/2}_{x\,i+1,j+1/2,k+1/2}\right),\\
\tilde{E}^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} &= E^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} + 2\sigma^{Ey}_{i+1/2,j+1/2,k+1/2}\left(E^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2} - E^{(+)n+1/2}_{y\,i+1/2,j,k+1/2}\right),\\
\tilde{E}^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2} &= E^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2} + 2\sigma^{Ey}_{i+1/2,j+1/2,k+1/2}\left(E^{(+)n+1/2}_{y\,i+1/2,j,k+1/2} - E^{(-)n+1/2}_{y\,i+1/2,j+1,k+1/2}\right),\\
\tilde{E}^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} &= E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} + 2\sigma^{Ez}_{i+1/2,j+1/2,k+1/2}\left(E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1} - E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k}\right),\\
\tilde{E}^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1} &= E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1} + 2\sigma^{Ez}_{i+1/2,j+1/2,k+1/2}\left(E^{(+)n+1/2}_{z\,i+1/2,j+1/2,k} - E^{(-)n+1/2}_{z\,i+1/2,j+1/2,k+1}\right),\\
\tilde{B}^{(+)n}_{x\,i+1/2,j,k} &= B^{(+)n}_{x\,i+1/2,j,k} + 2\sigma^{Bx}_{i,j,k}\left(B^{(-)n}_{x\,i+3/2,j,k} - B^{(+)n}_{x\,i+1/2,j,k}\right),\\
\tilde{B}^{(-)n}_{x\,i+3/2,j,k} &= B^{(-)n}_{x\,i+3/2,j,k} + 2\sigma^{Bx}_{i,j,k}\left(B^{(+)n}_{x\,i+1/2,j,k} - B^{(-)n}_{x\,i+3/2,j,k}\right),\\
\tilde{B}^{(+)n}_{y\,i,j+1/2,k} &= B^{(+)n}_{y\,i,j+1/2,k} + 2\sigma^{By}_{i,j,k}\left(B^{(-)n}_{y\,i,j+3/2,k} - B^{(+)n}_{y\,i,j+1/2,k}\right),\\
\tilde{B}^{(-)n}_{y\,i,j+3/2,k} &= B^{(-)n}_{y\,i,j+3/2,k} + 2\sigma^{By}_{i,j,k}\left(B^{(+)n}_{y\,i,j+1/2,k} - B^{(-)n}_{y\,i,j+3/2,k}\right),\\
\tilde{B}^{(+)n}_{z\,i,j,k+1/2} &= B^{(+)n}_{z\,i,j,k+1/2} + 2\sigma^{Bz}_{i,j,k}\left(B^{(-)n}_{z\,i,j,k+3/2} - B^{(+)n}_{z\,i,j,k+1/2}\right),\\
\tilde{B}^{(-)n}_{z\,i,j,k+3/2} &= B^{(-)n}_{z\,i,j,k+3/2} + 2\sigma^{Bz}_{i,j,k}\left(B^{(+)n}_{z\,i,j,k+1/2} - B^{(-)n}_{z\,i,j,k+3/2}\right).
\end{aligned}
$$

These formulas can be useful for modeling complex materials.

References

1. M. Castillejo, P. M. Ossi, and L. Zhigilei *Laser in Materials Science. Springer Series in Materials Science, Vol. 191.* (Springer, Cham, 2014).
2. N. M. Bulgakova, V. P. Zhukov, I. Mirza, et al., "Ultrashort-pulse laser processing of transparent materials: insight from numerical and semi-analytical models," in *Proc. SPIE 9735, Laser Applications in Microelectronic and Optoelectronic Manufacturing (LAMOM) XXI, San Francisco, California, United States, March 14, 2016* (SPIE, Vol. 9735, 2016), Article Number 97350N. doi 10.1117/12.2217585.
3. E. V. Zavedeev, V. V. Kononenko, V. M. Gololobov, and V. I. Konov, "Modeling the effect of fs light delocalization in Si bulk," Laser Phys. Lett. **11** (3), Article Number 036002 (2014). doi 10.1088/1612-2011/11/3/036002.

4. A. Rudenko, H. Ma, V. P. Veiko, et al., "On the role of nanopore formation and evolution in multi-pulse laser nanostructuring of glasses," Applied Physics A **124** (1), Article Number 63 (2018). doi 10.1007/s00339-017-1492-2.
5. F. Courvoisier, J. Zhang, M. Bhuyan, et al., "Applications of femtosecond Bessel beams to laser ablation," Applied physics. A, Materials science & processing **112**, 29–34 (2013). https://hal.science/hal-00845925v1. Cited April 28, 2026.
6. V. P. Zhukov and M. P. Fedoruk, "Numerically Implemented Impact of a Femtosecond Laser Pulse on Glass in the Approximation of Nonlinear Maxwell Equations," Mathematical Models and Computer Simulations **12** (1), 77–89 (2020). doi 10.1134/S207004822001010X.
7. S. Zhao and G. W. Wei, "High-order FDTD methods via derivative matching for Maxwell's equations with material interfaces," J. Comput. Phys. **200** (1), 60–103 (2004). doi 10.1016/j.jcp.2004.03.008.
8. S. Piperno, M. Remaki, and L. Fezoui, "A Nondiffusive Finite Volume Scheme for the Three-Dimensional Maxwell's Equations on Unstructured Meshes," SIAM J. Numer. Anal. **39** (6), 2089–2108 (2002). doi 10.1137/S0036142901387683.
9. V. P. Zhukov and M. P. Fedoruk, "Yee scheme with improved dispersion properties," Computer Optics (2026) in press.
10. K. S. Yee, "Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equations in Isotropic Media," IEEE Transactions on Antennas and Propagation **14** (3), 302–307 (1966). doi 10.1109/TAP.1966.1138693.

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