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LOW-FREQUENCY APPROXIMATION AND THE CHOICE OF OPTIMAL ELASTIC PARAMETERS FOR TWO-LAYER BLAST PROTECTION JACKETS

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Low-frequency approximation of elastic wave transmittance for two-layer structures is proposed. An accuracy of the approximation in a range up to several kilohertz is studied. Based on this approximation, the problem of the optimal elastic parameter choice for two-layer blast protection jackets is considered.

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1. Introduction. A blast wave produced by an explosion can result in lung injury characterized by diffuse contusions and haemorrhage of the alveoli [8]. The injury is caused by the wave's impingement upon the chest wall, rather than its propagation through the upper respiratory system [2]. The severity of the injury typically increases with the pressure amplitude of the blast wave and with the loading rate [4, 10]. Cumulative injury can be caused by repeated exposure to weaker blast events [7, 9].

Two of the authors of this paper (Winfree and Kang) have been involved in developing a jacket to protect against lung injury caused by blast waves impinging upon the chest. Our approach is that a jacket of one or more layers acts as a frequency filter, altering the wave transmitted into the chest by attenuating some frequencies of the incident wave and amplifying others [11]. This approach was also used by Cooper and colleagues [5]. Modeling all materials as linearly elastic, the transmission behavior through the jacket can be predicted analytically. Using this method, we consistently predicted that any two-layer design we considered would *amplify* across some range of low frequencies. This amplification is also apparent in Cooper's results. We began to suspect that it was impossible to find any two-layer combination that would not have this presumably undesirable response.

Other work we were involved in also required designing multi-layered structures as frequency filters for elastic or acoustic waves. Optimizing a layered media to achieve a particular transmission behavior is a difficult problem, but it has been much studied for thin-film coatings in optics. In the simplest cases, the propagation of elastic waves through layered media and the propagation of light through thin films are both described by the one-dimensional wave equation. Thus, the work of Winfree and Kang in elastic layered media led to a collaboration with the other authors of this paper (Tikhonravov and Trubetskov), experts in optimization methods for thin-film coatings. The goal of the collaboration was to develop software for the optimization of elastic layered media to achieve specified transmission or reflection behaviors in acoustics and elastic wave propagation.

The analysis presented here was conducted in the course of our collaboration. It supports previous findings that practical two-layer jackets (where practical means that the jacket must utilize reasonable materials and cannot be too heavy or thick) *do not* attenuate all low frequencies and indeed amplify some frequencies. This result holds if the following assumptions are met:

— all materials are linearly elastic and isotropic;

— both the input media and structure to be protected (in our case, respectively the air and the chest) can be modeled as half-spaces;

— all layers remain in contact, with no gaps opening between them;

— each layer of the protective structure is thin compared to the wavelength of a low-frequency harmonic wave in the material (we use $f = 2000$ Hz as an estimate for this frequency); this is typically the case because of bulk and weight constraints that limit the thickness of the layers;

— the characteristic material impedance of the structure to be protected (in our case, the chest) is greater than the impedance of the media from which the blast wave is incident; thus, the analysis can be applied to situations such as a blast wave impinging on a building or to a ship's hull subject to an underwater explosion; on the other hand, the analysis does *not* hold for divers subject to underwater explosions or noises, because the impedance of the person is similar to that of the water;

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— the characteristic material impedance of the layer closest to the structure is less than that of the structure, and the impedance of the outer layer is much greater than that of the incident media;

— the impedance of the layer farthest from the structure is less than the impedance of the structure (this would be a "soft" jacket).

While the last assumption seems very restrictive, in truth the analysis also holds for many cases when the impedance of the outermost layer is greater than that of the structure (for example, if the outer layer is metal and the structure is the chest); we just have not categorized the combinations of soft inner with stiff outer layers for which the analysis holds versus when it does not. In practice, we have never been able to devise any two-layer jacket subject to our weight and bulk constraints for which amplification is *not* predicted at low frequencies.

The analysis also demonstrates how a two-layer jacket can be optimized to achieve the greatest attenuation over a range of higher frequencies. As a particular example, we optimize a design — subject to constraints of a fixed total thickness and areal density — to achieve the maximum attenuation over 1 to 3 kHz. This frequency interval was chosen because Cooper and colleagues suggested that these are the components of an incident blast wave that are most responsible for lung injury [4, 5]. In brief, optimization requires that the outer layer be as heavy as allowed, while the inner layer should be a material with a low modulus and should be as thick as can be tolerated.

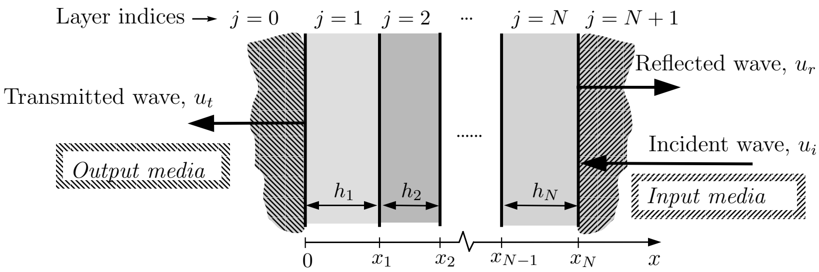

Fig. 1. Schematic of layered structure consisting of N layers. Properties of the input layer will be denoted by the index i (for "input"), but properties of other layers will be denoted by their index j. In this paper, we consider only a two-layer structure, so that $N = 2$

2. Basic equations for elastic waves in layered structures. Both the input and output media will be modeled as semi-infinite isotropic media (see Fig. 1). Waves propagate from the input to the output media across one or more layers considered to be isotropic and infinite in two dimensions. The waves are assumed to be longitudinal waves propagating normal to the interfaces.

The only component of displacement is in the x-direction; displacement is a function of the x-coordinate but not of the y- or z-coordinates. Therefore, denote displacement by $u = \widehat{u}(x,t)$. The deformation gradient γ and the velocity V are

$$\gamma = \partial u/\partial x\,, \tag{1}$$

$$V = \partial u/\partial t\,. \tag{2}$$

Assuming that body forces can be ignored, conservation of linear momentum reduces to

$$\frac{\partial \sigma}{\partial x} = \rho\,\frac{\partial V}{\partial t}\,, \tag{3}$$

where ρ is density and σ is the normal stress in the x-direction.

For linearly elastic isotropic solids, σ and γ are related by

$$\sigma = F\gamma\,, \tag{4}$$

where F is the *longitudinal modulus* of the material related to the more familiar moduli of isotropic materials by $F = \lambda + 2\mu = \dfrac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$. Here λ and μ are the Lamé moduli, E is Young's modulus, and ν is Poisson's ratio. The Lamé modulus μ is also known as the shear modulus.

Equations (1), (2), and (4) can be used to express the conservation of linear momentum in terms of u, resulting in the one-dimensional wave equation for displacement: $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$, where

$$c = \sqrt{F/\rho} \tag{5}$$

is the *longitudinal wave velocity* in the material.

Impedance z is defined by $z = \sigma/V$. In the case of a plane wave, the impedance can be shown to be $z = \pm\rho\, c$, where the choice of sign depends upon the direction of propagation of the wave. The quantity $\rho\, c$ is a material property called the *characteristic impedance* and will be denoted by an upper case Z:

$$Z = \rho\, c\,. \tag{6}$$

A common approach in studying the reflective and transmissive properties of layered structures is to perform analysis in the frequency domain [3]. This is done by considering the incident wave as a superposition of harmonic plane waves. Because all of the basic equations are linear, all waves in a layered structure can be considered as harmonic waves with the same frequency as that of the incident wave. This approach is equivalent to the application of the Fourier transform analysis to all time-dependent quantities of interest.

With the frequency-domain approach, it is important to note that computational formulas depend on the choice of sign in a time factor of the Fourier transform and on the choice of the direction of the normal to a layered structure (the choice of the direction of the x-axis in Figure 1). A detailed discussion of this subject can be found on the first pages of the book [6].

Using this approach, the incident particle velocity and stress are superpositions of harmonic waves V_i and σ_i, each of angular frequency ω and with the form

$$V_i(\omega, x, t) = V_I \exp\big(ik(x - x_N) + i\omega t\big)\,, \quad \sigma_i(\omega, x, t) = \sigma_I \exp\big(ik(x - x_N) + i\omega t\big)\,,$$

where V_I and σ_I are amplitudes of the waves, k is a wavenumber (its relation to other parameters is discussed later), and $i = \sqrt{-1}$ when it appears by itself (as opposed to its use as a subscript to indicate a property of the incident media). The dependence on x in the form $ik(x - x_N)$ is taken here for convenience, so that V_I and σ_I are amplitudes at the boundary between the input media and the layered structure (see Figure 1). A similar expression can be written for the incident displacement u_i, but we will not need it.

Similarly, the transmitted waves are superpositions of harmonic waves of the forms

$$V_t(\omega, x, t) = V_T \exp\,(ikx + i\omega t)\,, \tag{7}$$

$$\sigma_t(\omega, x, t) = \sigma_T \exp\,(ikx + i\omega t)\,. \tag{8}$$

Here V_T and σ_T are amplitudes of the transmitted waves at the boundary of the output medium and the layered structure.

Finally, the reflected waves are superimposed of harmonic waves that propagate in the opposite direction to the incident and transmitted waves:

$$V_r(\omega, x, t) = V_R \exp\big(ik(x - x_N) - i\omega t\big)\,, \quad \sigma_r(\omega, x, t) = \sigma_R \exp\big(ik(x - x_N) - i\omega t\big)\,.$$

Again, the dependence on x is specified in a form that corresponds to the amplitudes of reflected waves taken at the boundary between the input media and the layered structure.

Our goal is to find the *stress transmission coefficient* T and *stress reflection coefficient* R of the layered structure, where we use the definitions

$$T = \sigma_T/\sigma_I\,, \quad R = \sigma_R/\sigma_I\,. \tag{9}$$

In the following, we use two different recurrent techniques for calculating the transmission and reflection coefficients T and R of a layered structure, the so-called *matrix technique* and the *admittance technique*. To show consistency between these techniques, the following subsections provide a concise description of major steps of the derivations for the reflection and transmission coefficients.

2.1. Solution by the matrix technique. We restate Eq. (3) for convenience:

$$\frac{\partial \sigma}{\partial x} = \rho\,\frac{\partial V}{\partial t}\,, \tag{10}$$

This is a first order partial differential equation for stress σ and velocity V. Another first-order equation is obtained by differentiating Eq. (4) with respect to time and making use of Eqs. (1) and (2) to eliminate γ, resulting in

$$\frac{\partial \sigma}{\partial t} = F\,\frac{\partial V}{\partial x}\,. \tag{11}$$

In Eqs. (10) and (11), V and σ are functions of two variables, namely x and t. We transit to the frequency domain by representing them as a superposition of the terms $V(x)\exp(i\omega t)$ and $\sigma(x)\exp(i\omega t)$ with various angular frequencies ω. In fact, each pair of functions $V(x)$ and $\sigma(x)$ depends on the parameter ω, but, as it is usually done, we omit the direct indication of the dependence of V and σ on ω for conciseness.

Equations (10) and (11) give the system of ordinary differential equations for all pairs of these functions:

$$\frac{d\sigma}{dx} = i\omega\rho V\,, \qquad \frac{dV}{dx} = \frac{i\omega}{\rho c^2}\,\sigma\,. \tag{12}$$

Here the wave speed c is given by Eq. (5).

Solutions of Eqs. (12) inside the jth layer can be written in the form

$$\begin{aligned} \sigma(x) &= \sigma(x_{j-1})\cos\frac{\omega}{c_j}(x - x_{j-1}) + i\,\rho_j c_j\,V(x_{j-1})\sin\frac{\omega}{c_j}(x - x_{j-1})\,, \\ V(x) &= \frac{i}{\rho_j c_j}\,\sigma(x_{j-1})\sin\frac{\omega}{c_j}(x - x_{j-1}) + V(x_{j-1})\cos\frac{\omega}{c_j}(x - x_{j-1})\,, \end{aligned} \tag{13}$$

where ρ_j and c_j are the density and wave velocity of the material of layer j and x_{j-1} is the coordinate of the leftmost boundary of this layer in Figure 1.

Now put $x = x_j$ in Eqs. (13), so that we have $x_j - x_{j-1} = h_j$ on the right-hand side of these equations (h_j is the thickness of the jth layer). In this case, Eqs. (13) describe how the pair $(\sigma,\, V)$ is transformed from the left boundary to the right boundary of the jth layer. We introduce the value $\varphi_j = \dfrac{\omega}{c_j}\,h_j$, which will be called the *phase thickness* of the jth layer. We can write the above-mentioned transformation in the matrix form

$$\begin{pmatrix}\sigma\\ V\end{pmatrix}_{x=x_j} = \begin{pmatrix}\cos\varphi_j & i\,Z_j\,\sin\varphi_j\\ \dfrac{i}{Z_j}\,\sin\varphi_j & \cos\varphi_j\end{pmatrix}\begin{pmatrix}\sigma\\ V\end{pmatrix}_{x=x_{j-1}}, \tag{14}$$

where Z_j is the characteristic impedance of the jth layer as defined by Eq. (6).

By analogy with optics, the matrix on the right-hand side of Eq. (14) can be called the characteristic matrix of the jth layer [6].

Provided the layers remain in contact, the functions $\sigma(x)$ and $V(x)$ are continuous across the layer boundaries: $\sigma(x_j^+) = \sigma(x_j^-)$ and $V(x_j^+) = V(x_j^-)$. Thus, the characteristic matrices enable the recursive calculation of these functions, working from the leftmost boundary between the output medium and the first layer to the rightmost boundary between the last layer and the input medium in Figure 1. As a result, we have

$$\begin{pmatrix}\sigma\\ V\end{pmatrix}_{x=x_N} = M\begin{pmatrix}\sigma\\ V\end{pmatrix}_{x=0}, \tag{15}$$

where M is the product of characteristic matrices of all layers:

$$M = \prod_{j=N}^{1}\begin{pmatrix}\cos\varphi_j & i\,Z_j\,\sin\varphi_j\\ \dfrac{i}{Z_j}\,\sin\varphi_j & \cos\varphi_j\end{pmatrix}. \tag{16}$$

At the leftmost and rightmost boundaries of the layered structure we have

$$\sigma\big|_{x=0} = \sigma_T\,, \quad V\big|_{x=0} = V_T\,, \quad \sigma\big|_{x=x_N} = \sigma_I + \sigma_R\,, \quad V\big|_{x=x_N} = V_I + V_R\,, \tag{17}$$

where the indices I, T, and R are used to denote quantities related to the incident, transmitted, and reflected waves and taken at the respective boundaries of the layered structure.

Now we need a relation between stress and velocity in the output media. Substituting Eqs. (7) and (8) into Eqs. (12) gives $k\sigma_T = \omega\rho_0 V_T$, $kV_T = \dfrac{\omega}{\rho_0 c_0^2}\,\sigma_T$, where ρ_0 and c_0 are the density and longitudinal wave velocity of the output medium. From these we have that $k = \omega/c_0$ in the output medium and

$$\sigma_T = Z_0 V_T\,, \tag{18}$$

where $Z_0 = \rho_0 c_0$ is the characteristic impedance of the output medium.

Using the subscript "i" to denote elastic parameters of the input medium, we similarly obtain that $k = \omega/c_i$ in the input medium and

$$\sigma_I = Z_i V_I\,, \quad \sigma_R = -Z_i V_R\,, \tag{19}$$

where $Z_i = \rho_i c_i$ is the characteristic impedance of the input medium.

Equations (17), (18), and (19) together with the matrix equation (15) allow us to express the transmission and reflection coefficients (9) through the elements of the matrix M. We will denote these elements as m_{11}, m_{12}, m_{21}, and m_{22}. Expressions for T and R are as follows:

$$T = \frac{2}{m_{11} + m_{12}/Z_0 + m_{21}Z_i + m_{22}Z_i/Z_0}\,, \tag{20}$$

$$R = \frac{m_{11} + m_{12}/Z_0 - m_{21}Z_i - m_{22}Z_i/Z_0}{m_{11} + m_{12}/Z_0 + m_{21}Z_i + m_{22}Z_i/Z_0}\,. \tag{21}$$

Expressions (20) and (21) together with Eq. (16) are basic formulas of the matrix technique. In the case when there are no layers between the input and output media, we have $m_{11} = m_{22} = 1$, $m_{12} = m_{21} = 0$, and Eqs. (20) and (21) are reduced to the expressions presenting the transmission and reflection coefficients of the boundary between input and output media:

$$T = \frac{2Z_0}{Z_0 + Z_i}\,, \quad R = \frac{Z_0 - Z_i}{Z_0 + Z_i}\,. \tag{22}$$

2.2. Solution by the admittance technique. Now consider the admittance technique. It will be seen that it is similar to the impedance recurrent technique presented in Brekhovskikh's book [3]. We introduce *admittance* A as the ratio of particle velocity to stress: $A = 1/z = V/\sigma$.

The admittance depends on the x-coordinate and it is a continuous function of x inside the layered structure, including the boundaries between the layers. This follows from the continuity of V and σ. Taking Eqs. (13) at $x = x_j$ and dividing the second equation by the first one, we obtain

$$A(x_j) = \frac{A(x_{j-1}) + i/Z_j\ \tan\varphi_j}{1 + iZ_j\ A(x_{j-1})\tan\varphi_j}\,. \tag{23}$$

This formula gives a recurrent algorithm for calculating the admittance from the leftmost boundary of the layered structure to its rightmost boundary. According to Eqs. (17) and (18), at the left boundary

$$A(0) = 1/Z_0\,. \tag{24}$$

This equation gives the initial condition for the recurrent formula (23).

Using Eqs. (9), (17), and (19), it is easy to obtain that at the right boundary of the layered structure

$$A(x_N) = \frac{1}{Z_i}\,\frac{1-R}{1+R}\,. \tag{25}$$

Solving Eq. (25) with respect to R, we come to the equation expressing the amplitude reflection coefficient R through the admittance $A(x_N)$, which is often called the input admittance:

$$R = \frac{1 - Z_i\,A(x_N)}{1 + Z_i\,A(x_N)}\,. \tag{26}$$

Equations (23), (24), and (26) are basic formulas of the admittance technique providing a recurrent algorithm for calculating R. The amplitude transmission coefficient T can also be calculated using the admittance technique. Here we provide only the final formula for T without details of its derivation:

$$T = \frac{2/Z_i}{1/Z_0 + 1/Z_1}\prod_{j=1}^{N}\frac{A(x_j) + 1/Z_j}{A(x_j) + 1/Z_{j+1}}\ \exp(-i\varphi_j)\,.$$

3. Low-frequency approximation for the intensity transmission coefficient of a two-layer structure. In this section, instead of the angular frequency ω we will use the frequency $f = \dfrac{\omega}{2\pi}$, which is a more common parameter in many practical applications.

The transmission coefficient $T(f)$ of a two-layer structure is calculated by the formula (20) through the elements of the matrix product (16), where $N = 2$ and

$$\varphi_{1,2} = \frac{2\pi f h_{1,2}}{c_{1,2}} = \frac{2\pi h_{1,2}}{\lambda_{1,2}} \,. \tag{27}$$

Recall that $h_{1,2}$, $c_{1,2}$, and $\rho_{1,2}$ are thicknesses, longitudinal velocities, and material densities of the first and second layers of the structure. We have introduced $\lambda = c/f$, which is the wavelength of a wave of frequency f in a material with wave speed c.

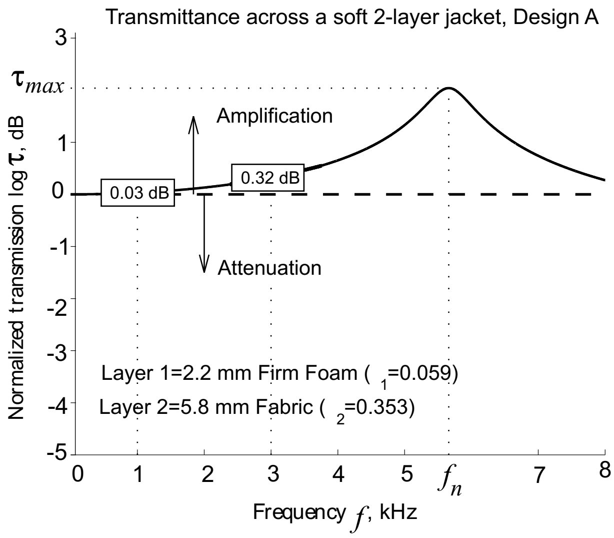

Fig. 2. Exact normalized transmittance predicted for the initial design of a soft two-layer jacket (this is called "Design A" in the figure title). The layer materials and their thicknesses are provided in the figure. The input media is always air and the output media is always water (meant to represent the chest). The vertical scale is $\log_{10} \tau(f)$. The labels at $f = 1$ and 3 kHz indicate $\tau(f)$ at these frequencies: they are a convenience in evaluating the performance of the design over the 1 to 3 kHz target range. Note that this design *amplified* the transmission of the incident wave over this band and in fact provides no attenuation at any frequency covered in this plot. More of the energy in the incident blast wave over these frequencies is passed into the chest than if *no* jacket were worn!

Let us introduce the normalized intensity transmission coefficient by the equation $\tau(f) = \left|\frac{T(f)}{T(0)}\right|^2$. For conciseness, this may simply be called the *normalized transmittance.*

At $f = 0$, the matrix elements m_{11} and m_{22} equal 1, while the matrix elements m_{12} and m_{21} are zeroes. Thus, it follows from Eq. (20) that $T(0)$ equals the transmission coefficient across the *uncovered* interface of the input and output media (see Eqs. (22)). Thus, the *normalized transmittance is the ratio of the intensity transmission coefficients when the layers are used versus when they are not*: this is a measure of how much the layers increase the transmission ($\tau > 1$) or decrease the transmission ($\tau < 1$) compared to having no layers between the incident and output media.

Figure 2 presents a plot of normalized transmittance calculated from Eq. (20) for a soft two-layer blast protection covering at the boundary between air and the chest wall (modeled as a half-space with the wave speed and density of water). The layer closest to the chest is a "firm" foam, and the second layer is fabric. The material properties used are presented in the table. The thicknesses of the layers were chosen to give a total thickness of 8 mm and a total areal density of 0.49 g/cm^2, values similar to those for a soft ballistic jacket. The materials and layer thicknesses for this and all examples to follow are provided in each figure. Also provided in each figure are the phase thicknesses φ_1 and φ_2 for a harmonic wave of frequency $f = 2000$ Hz. Because the jacket layers are thin compared to a wavelength $\lambda = c/f$ at this frequency, φ_1 and φ_2 are small values for all of our examples and we can replace trigonometric functions of them with approximations valid for small quantities. These approximations will be crucial in the analysis.

Material parameters used for calculations, arranged in order of decreasing modulus F. Also shown are wavelengths $\lambda = c/f$ at $f = 2$ kHz. Values for copper, water, and air are from Appendix B of [1]. The fabric is a ballistic fabric: its density and wave speed across several fabric layers were measured at Dominca. Wave speed was measured by finding the transit time across 32 layers of the fabric, total thickness approximately 6 mm, using ultrasonic transducers with center frequencies of 500 kHz. The foams are two different crosslinked, closed-cell polyolefine foams: their wave speeds and densities were measured at Dominca

Material	ρ g/cm^3	c km/s	$Z = \rho c$ MPa/(m/s)	$F = \rho c^2$ GPa	λ, mm at $f = 2$ kHz
Copper	8.93	4.99	44.6	222	2495
Water	1.10	1.4	1.54	2.156	700
Fabric	0.80	0.208	0.167	0.0346	104
Firm Foam	0.0961	0.4590	0.0441	0.0202	230
Soft Foam	0.0320	0.2910	0.0093	0.0027	146
Air	0.0013	0.329	0.000428	0.00014	165

The frequency dependence presented in Figure 2 has some special features. At low frequencies, the normalized transmittance τ increases with frequency. It reaches its first maximum $\tau_{\max}$ at $f = f_n$ and decreases at $f > f_n$, the subscript n being chosen to indicate that this is a natural or resonance frequency. There are other maxima at frequencies higher than those plotted in Figure 2.

Assume now that we wish to maximize the attenuation over 1 to 3 kHz. The design of Figure 2 *amplifies* these components of the transmitted signal. If we are to block rather than enhance transmission over these frequencies, we need to move f_n to a frequency below 1 kHz. To accomplish this, we derive an approximate expression for $\tau(f)$ that will be easier to understand.

As indicated in Figure 2, the phase thicknesses $\varphi_{1,2}$ are small at $f = 2$ kHz (the center of our frequency band of interest). Thus, we shall consider an approximate expression for the normalized transmittance obtained by using the first terms of trigonometric function expansions in the layer characteristic matrices M_j. Let us write the characteristic matrices of the first and second layer in the forms

$$M_1 \approx \begin{pmatrix} 1-\varphi_1^2/2 & iZ_1\,(\varphi_1-\varphi_1^3/6) \\ (i/Z_1)\,(\varphi_1-\varphi_1^3/6) & 1-\varphi_1^2/2 \end{pmatrix}, \quad M_2 \approx \begin{pmatrix} 1-\varphi_2^2/2 & iZ_2\,(\varphi_2-\varphi_2^3/6) \\ (i/Z_2)\,(\varphi_2-\varphi_2^3/6) & 1-\varphi_2^2/2 \end{pmatrix}.$$

The product of these matrices gives the matrix M whose elements are used in the expression (20) for T. Recalling that the phases φ_1 and φ_2 are proportional to the frequency f (see Eq. (27)), it can be seen that the matrix elements m_{11} and m_{22} are real and are expressed as series of terms with even powers of frequency f, while the matrix elements m_{12} and m_{21} are imaginary and are represented as series with odd powers of f. To calculate $\tau(f)$, one should take a squared modulus of T from which we conclude that the denominator of $\tau(f)$ is a series of terms with even powers of f. From the definition of the normalized transmittance, we have $\tau(0) = 1$. On the whole, this means that at low frequencies the normalized transmittance $\tau(f)$ can be represented in the form

$$\tau(f) \approx \frac{1}{1+\alpha f^2+\beta f^4}\,. \tag{28}$$

We shall call this form a low-frequency approximation of the normalized transmittance.

The coefficients α and β in Eq. (28) have been expressed as functions of the properties of the layers in the appendix and some constraints that ensure that $\alpha < 0$ are also discussed there. In the following, we will only need to work with Eq. (28) in its current form in order to make some practically useful conclusions.

If the coefficient α in Eq. (28) is negative, then $\tau(f)$ is a monotonically increasing function of f at low frequencies: the jacket *amplifies* the transmission of low frequencies. This is just as observed in Figure 2.

According to Eq. (28), the maximum value of $\tau(f)$ is attained at

$$f_n \approx \sqrt{-\frac{\alpha}{2\beta}} \tag{29}$$

and this value is

$$\tau_{\max} \approx \frac{1}{1-\alpha^2/(4\beta)}\,. \tag{30}$$

Equations (29) and (30) allow the parameters α and β to be expressed by the values f_n and $\tau_{\max}$. We have that $\alpha = -\dfrac{2}{f_n^2}\,\dfrac{\tau_{\max}-1}{\tau_{\max}}$, $\beta = \dfrac{1}{f_n^4}\,\dfrac{\tau_{\max}-1}{\tau_{\max}}$.

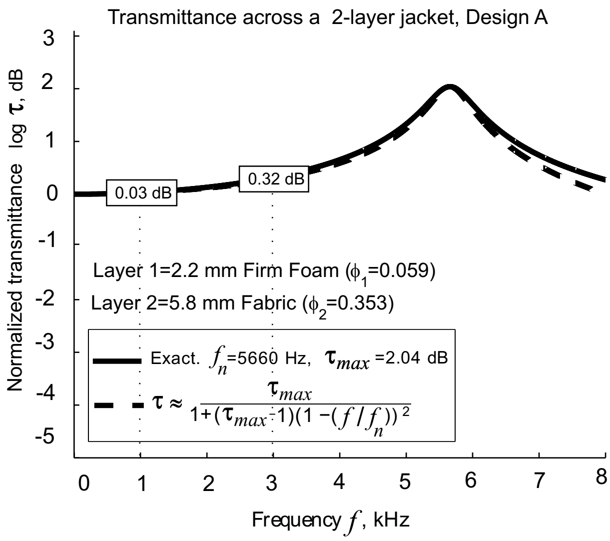

Fig 3. The approximation of Eq. (31) compared with the exact plot of $\tau(f)$. In the approximation, we take $\tau_{\max}$ and f_n from the exact solution

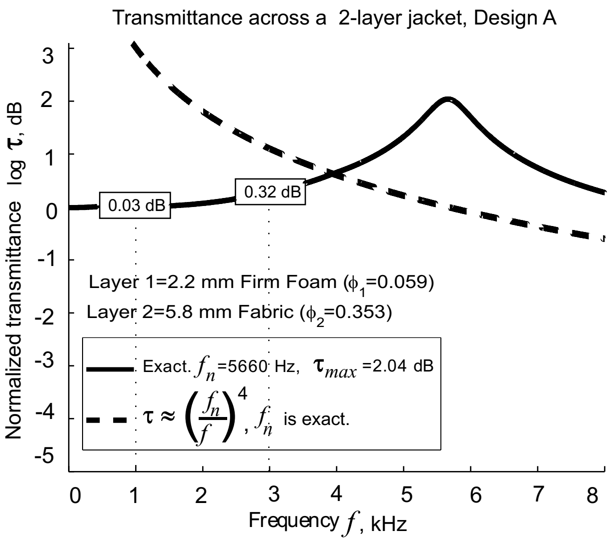

Fig 4. The approximation $\tau(f) \approx \left(\dfrac{f_n}{f}\right)^4$ from Eq. (32) compared with the exact plot of $\tau(f)$. In the approximation, we take f_n from the exact solution

Using these equations, we can rewrite Eq. (28) in the form

$$\tau(f) \approx \frac{\tau_{\max}}{1+(\tau_{\max}-1)\big[1-(f/f_n)^2\big]^2}\,. \tag{31}$$

Figure 3 compares the exact normalized transmittance from Figure 2 with its low-frequency approximation calculated in accordance with Eq. (31). The agreement is good for $f < f_n$ but less appropriate for higher frequencies. If we cared to pursue a better approximation, we would include at least the f^6 terms in the denominator of Eq. (28).

In this and all of our examples, $\tau_{\max} \gg 1$. Thus, we will further approximate that $\alpha \approx -2/f_n^2$ and $\beta \approx 1/f_n^4$. For $f > f_n$, the last term in the denominator of Eq. (28) plays a dominant role, and we can further approximate

$$\tau(f) \approx \left(\frac{f_n}{f}\right)^4 \quad \text{for } f > f_n\,. \tag{32}$$

This approximation is presented by the dashed curve in Figure 4. Though this approximation is truly a poor description of τ for this jacket concept, it will be useful nonetheless in modifying the design and it will become an increasingly better match to τ for $f > f_n$ as we do so.

It is worthwhile to note that the low-frequency approximation considered in this section is valid not only for two-layer structures but also for arbitrary multilayer structures, providing that the sum of phase thicknesses of all layers is small. This follows from the fact that considerations of this section are independent of the number of structure layers.

4. Considerations on the optimal choice of layer parameters. In this section we will optimize a two-layer jacket design subject to the constraints that the total thickness and areal density must remain fixed at 8 mm and 0.49 g/cm^2, respectively. The materials we will consider for the jacket are described in the table.

It will be useful to define the following ratios between the impedance of the output media and the impedances of each of the other materials:

$$n_1 = \frac{Z_0}{Z_1}, \quad n_2 = \frac{Z_0}{Z_2}, \quad n_i = \frac{Z_0}{Z_i}. \tag{33}$$

The methods in this section are based on the admittance recurrent technique described in Section 2.2. We will suppose that the layer phase thicknesses φ_1 and φ_2 are small enough, so that we can use the first order approximations for $\tan\varphi_{1,2}$, i.e., we can put $\tan\varphi_1 = \varphi_1$ and $\tan\varphi_2 = \varphi_2$ in Eq. (23). For the first layer, Eq. (23) together with the definition of n_1 from Eq. (33) gives

$$A(x_1) = \frac{(1/Z_0) + i\varphi_1/Z_1}{1 + i\varphi_1/n_1}. \tag{34}$$

Minimizing transmittance requires that the output medium and the first layer have a high contrast in impedances. According to this supposition, $n_1 \gg 1$ and we will ignore the term φ_1/n_1 in the denominator of Eq. (34), leaving

$$Z_0\, A(x_1) = 1 + in_1\varphi_1. \tag{35}$$

For the second layer, Eq. (23) gives $A(x_2) = \dfrac{A(x_1) + i\varphi_2/Z_2}{1 + iA(x_1)\, Z_2\, \varphi_2}$. Using Eq. (35) along with the definition of n_2 from Eq. (33), we can rewrite this as follows:

$$Z_0\, A(x_2) = \frac{1 + in_1\varphi_1 + in_2\varphi_2}{1 + i\,(1 + in_1\varphi_1)\, \varphi_2/n_2}. \tag{36}$$

Equation (26) allows the amplitude reflection coefficient R to be expressed through the admittance $A(x_2)$. Using the definition of n_i from Eq. (33), we can write the expression for R in the following form:

$$R = \frac{n_i - Z_0\, A(x_2)}{n_i + Z_0\, A(x_2)}. \tag{37}$$

If we assume that there are no energy losses in the layered structure, then $(Z_i/Z_0)|T|^2 = 1 - |R|^2$ and we have from Eq. (37) that

$$|T|^2 = \frac{4n_i^2\, \mathrm{Re}\{Z_0\, A(x_2)\}}{\Big[n_i + \mathrm{Re}\{Z_0\, A(x_2)\}\Big]^2 + \Big[\mathrm{Im}\{Z_0\, A(x_2)\}\Big]^2}. \tag{38}$$

It is seen now that to find the normalized transmittance $\tau(f)$, one should separate the real and imaginary parts of the admittance $A(x_2)$. It follows from Eq. (36) that

$$\mathrm{Re}\{Z_0\, A(x_2)\} = \frac{1 + \varphi_2^2}{\big(1 - (n_1/n_2)\varphi_1\varphi_2\big)^2 + (\varphi_2/n_2)^2}, \tag{39}$$

$$\mathrm{Im}\{Z_0\, A(x_2)\} = \frac{(n_1\varphi_1 + n_2\varphi_2)\big(1 - (n_1/n_2)\varphi_1\varphi_2\big) - \varphi_2/n_2}{\big(1 - (n_1/n_2)\varphi_1\varphi_2\big)^2 + (\varphi_2/n_2)^2}. \tag{40}$$

We will assume that the impedance of the output medium is significantly greater than the impedance of the input medium, i.e., that $n_i \gg 1$ in Eq. (38). For $f = 0$ we have from Eq. (36) that $Z_0\, A(x_2) = 1$ and Eq. (38) gives $\big|T(0)\big|^2 = 4n_i^2/(n_i + 1)^2 \approx 4$.

Obviously, $|T|^2$ will be much greater than this value when $\mathrm{Re}\{Z_0\, A(x_2)\}$ is much greater than 1 and $\mathrm{Im}\{Z_0\, A(x_2)\}$ is still a relatively small value. At low frequencies, $\mathrm{Re}\{Z_0\, A(x_2)\}$ will be great when the first term in the denominator of Eq. (39) equals to zero. Thus, it is natural to suppose that $\tau(f) = |T(f)|^2/|T(0)|^2$ is close to its maximum value when

$$(n_1/n_2)\,\varphi_1\varphi_2 = 1\,. \tag{41}$$

Substituting here φ_1, φ_2 from Eq. (27) and n_1, n_2 from Eq. (33), we find that Eq. (41) is satisfied when

$$f = f_n = \frac{c_1}{2\pi}\sqrt{\frac{\rho_1}{\rho_2 h_1 h_2}} = \frac{1}{2\pi}\frac{\sqrt{F_1/h_1}}{\sqrt{\rho_2 h_2}} = \frac{1}{2\pi}\sqrt{\frac{k_1}{m_2}}\,, \tag{42}$$

where $m_2 = \rho_2 h_2$ is the mass per unit area of the outermost layer and $k_1 = \sqrt{F_1/h_1}$ is the "spring constant" per unit area of the innermost layer. Thus, our assumptions have led us to a lumped-parameter model: Eq. (42) is the equation for the natural frequency f_n of a point mass m_2 connected to a fixed wall by a massless spring with spring constant k_1. Though this approximation will not be particularly good for our first jacket design, we can still use it to understand how to move f_n to lower frequencies and decrease the transmission over some range of frequencies greater than f_n.

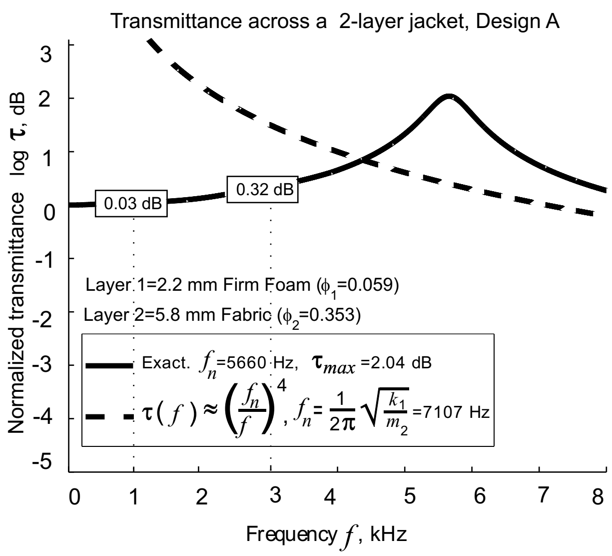

Fig 5. The approximation $\tau(f) \approx \left(\frac{f_n}{f}\right)^4$ from Eq. (32) compared with the exact plot of $\tau(f)$. In the approximation, we take $f_n = \frac{1}{2\pi}\sqrt{\frac{k_1}{m_2}}$ from Eq. (42)

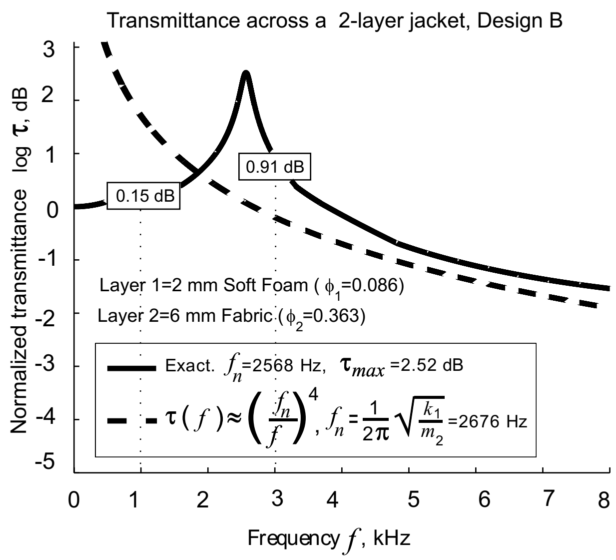

Fig 6. As in Figure 5, but the design has been revised so that a softer foam is used behind the fabric

In Figure 5, we approximate f_n by this equation and plot the approximation $\tau(f) = (f_n/f)^4$ for the design considered in Figures 2 to 4. We find that Eq. (42) predicts that $f_n = 7107$ Hz, while the maximum value of $\tau(f)$ is really attained at 5650 Hz (see Figure 2). The discrepancy indicates that describing the outer shell as a point mass and the inner shell as a massless spring is not adequate for this design. Of course, the curve $\tau(f) = (f_n/f)^4$ remains a poor description of τ.

Though not accurate for this design, Eq. (42) still provides useful guidance in how to modify the design to achieve lower transmittance across our 1 to 3 kHz target. The first step is to force the resonance frequency f_n below our target range. By inspection of Eq. (42), we see that this can be done by increasing the total mass in the outer shell, by selecting a lower-modulus material for the inner shell, or by increasing the thickness of the inner layer.

In Figure 6, a softer, less dense foam has been used for the inner layer. To maintain the same areal density and thickness as in the first design, the outer layer has become slightly thicker and the inner layer slightly thinner. The result is that f_n has indeed decreased to 2568 Hz. If predicted by Eq. (42), $f_n = 2676$ Hz, a significantly improved match compared to that for the prediction for the previous design. Overall, however, the approximation $\tau(f) = (f_n/f)^4$ for f_n is still not very good. Worse, the peak transmittance now falls within the range over which we aim to decrease the transmittance!

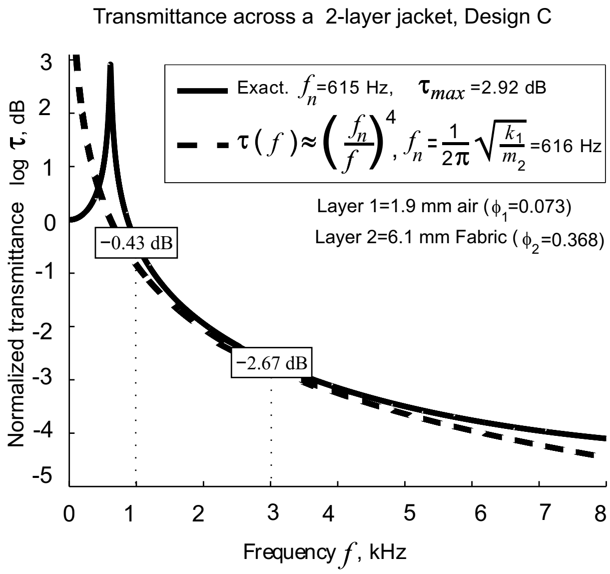

Fig 7. As in Figure 5, but air is used instead of foam behind the fabric

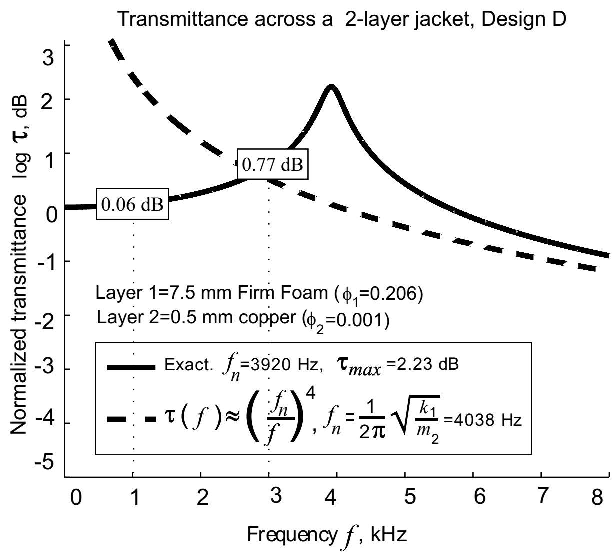

Fig 8. In this and the remaining figures, the outer layer of the jacket is copper instead of fabric. Here it is backed by the firm foam of Figure 5. Compared to that figure, we have been able to move f_n to a lower frequency, but there is still no attenuation over our 1 to 3 kHz target band

Continuing, we replace the inner layer by air. Again, the outer layer becomes a little thicker, and the inner layer a little thinner if we hold the total mass and thickness constant. The predicted transmissivity is plotted in Figure 7. Note that the approximation $\tau(f) = (f_n/f)^4$ has improved for $f > f_n$ and is a reasonable match over much of our 1 to 3 kHz range of interest. Our lumped-mass approximation for the natural frequency has also improved: Eq. (42) predicts that $f_n = 613$ Hz, a very good match to the exact value of $f_n = 612$ Hz. By moving the natural frequency to a lower value, we have finally achieved attenuation over our 1 to 3 kHz target range.

A problem with all of these designs is that the inner layer is so thin that, in reality, we might expect it to “bottom out,” violating our assumption of linear elasticity. With the mass and thickness constraints we are enforcing, nearly the entire jacket must be fabric, leaving little thickness available for the inner layer. In addition, we may be able to better block the transmission across our target range if we can increase the thickness of the inner layer, effectively reducing the spring constant k_1.

To increase the thickness of the inner layer, let us consider using a more dense material for the outer layer. We will choose copper and pair it in turn with the firm foam, the soft foam, and the air that we have just considered, still fixing the areal density and total thickness. This lets us replace the six or so millimeters of fabric by only 0.5 mm of copper. The predicted exact transmittance is shown for each combination in Figures 8 to 10 along with the approximation of Eq. (32) for $\tau(f)$, using f_n computed from Eq. 42. The approximation of $\tau(f)$ for frequencies just above the first resonance remains poor when the firm foam is used, but is better when the soft foam or air is used, situations for which the lumped-parameter model becomes more appropriate. Again, the usefulness of the approximation is in systematically guiding the design modifications in order to achieve additional attenuation over the target range of frequencies.

Of the three designs with copper, only the copper backed by air provides significant attenuation over the entire target range. Indeed, copper paired with the firm foam amplifies the 1−3 kHz components of the transmitted signal. When the copper is backed by the soft foam, there is amplification up to around 2 kHz and then some attenuation for frequencies greater than this.

5. Discussion. In this paper we examined the transmission behavior of a two-layer jacket on the chest using a well-established method of describing the transmission of waves through layered media. The method is often applied in acoustics and is closely related to methods used in optics and even in transmission line theory.

We provided rigorous and self-consistent derivations of two recurrent techniques (the matrix technique and

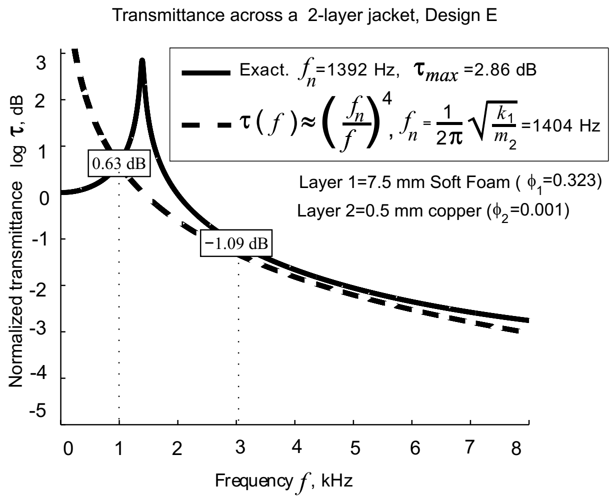

Fig 9. As in the Figure 8, but a softer foam is used here

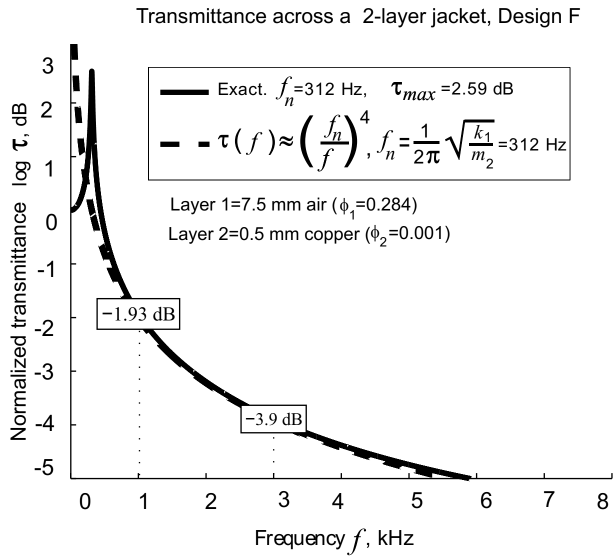

Fig 10. As in Figure 8, but air is used instead of foam behind the copper

the admittance technique) for calculating reflection and transmission properties of elastic structures.

We find that it is difficult to design a jacket of moderate weight and bulk that effectively blocks the transmission of incident blast energy over frequencies of 1 to 3 kHz. In fact, it can be shown that, in the absence of dissipation, a soft two-layer jacket (where soft means that the impedance of the outer layer is less than that of the chest and the impedance of the inner layer is intermediate to that of the chest and the air) is guaranteed to amplify some range of low frequencies and that attenuation is possible only for frequencies above this range. The same holds true for some combinations of “hard” jackets (meaning the impedance of the outer layer is greater than that of the chest), but it is harder to specify what these combinations are.

If we consider the peak amplification to occur at a natural frequency f_n, the task then is to push f_n to lower frequencies. We derived an approximation for the transmission behavior of the jacket at frequencies $f > f_n$ and also an approximation for f_n that turns out to be identical to the natural frequency predicted by a lumped-parameter model. Neither approximation is particularly good for most of our jacket designs, but the insight we gain from them allows us to systematically vary the design in order to push f_n to a lower frequency. We find that, given the constraints we apply and the materials we consider in this paper, the only designs that block the transmission of energy over our entire target frequency range are those with an inner layer of air.

Our interest was in blocking the transmission of energy for frequencies of 1 to 3 kHz. We have not considered the ramifications of amplifying the transmission of the low frequencies, but they could be quite serious. Note, however, that because we are using a continuum model, the predicted transmission remains finite in the absence of dissipation. In contrast, a lumped-parameter model would predict an infinite response at resonance unless dissipation were added.

Unfortunately, we find that — in the absence of anisotropy or any non-linear behaviors — amplification of low frequencies is unavoidable for any “soft” two-layer jacket and for many two-layer jackets with a hard outer shell (and very possibly for all such jackets). Our models can readily include dissipation by using complex moduli to describe dissipative materials. Frequency-dependent material behavior can be modeled by allowing the moduli to be functions of frequency. Anisotropic behaviors can also be handled: in fact, these behaviors are important in non-destructive testing and ultrasonic imaging.

Though we systematically optimized the design of a two-layer jacket, optimization becomes more difficult as more layers are added or if nonlinear behaviors are modeled. For these cases, we find that optimization methods implemented computationally become valuable.

We caution that, in reality, the chest probably should not be modeled as a half-space. Better approximations would consider the chest wall to be a layer with the appropriate thickness and might also include the lungs and even the opposite wall of the torso.

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Appendix. The coefficients α and β in Eq. (28) can be derived. They are

$$\alpha = \frac{-4\pi^2}{\big(c_1 c_2 Z_1 Z_2 (Z_0 + Z_i)\big)^2} \big(c_1^2 h_2^2 Z_1^2 (Z_0^2 - Z_2^2)(Z_2^2 - Z_i^2) + 2c_1 c_2 h_1 h_2 Z_1 Z_2 (Z_0^2 - Z_1^2)(Z_2^2 - Z_i^2) + c_2^2 h_1^2 Z_2^2 (Z_0^2 - Z_1^2)(Z_1^2 - Z_i^2)\big) \tag{43}$$

and

$$\beta = \frac{4\pi^4}{3\big(c_1^2 c_2^2 Z_1 Z_2 (Z_0 + Z_i)\big)^2} \Big(c_2^4 h_1^4 Z_2^2 \big(-4Z_1^4 + 3Z_i^2 Z_1^2 - 2Z_0 Z_i Z_1^2 + Z_0^2(3Z_1^2 - 4Z_i^2)\big) + 16 c_1 c_2^3 h_1^3 h_2 Z_1 Z_2 (Z_0^2 - Z_1^2)(Z_2^2 - Z_i^2) + 12 c_1^2 c_2^2 h_1^2 h_2^2 (Z_0^2 - Z_1^2)(Z_1^2 + Z_2^2)(Z_2^2 - Z_i^2) - 16 c_1^3 c_2 h_1 h_2^3 Z_1 Z_2 (Z_1^2 - Z_0^2)(Z_2^2 - Z_i^2) + c_1^4 h_2^4 Z_1^2 \big(-4Z_2^4 + 3Z_i^2 Z_2^2 - 2Z_0 Z_i Z_2^2 + Z_0^2(3Z_2^2 - 4Z_i^2)\big)\Big). \tag{44}$$

For the case of a blast wave in the air impinging on the chest covered by a two-layer jacket, it is appropriate to assume that

$$Z_0 \gg Z_i\,, \quad Z_1 \geqslant Z_i\,, \quad Z_2 \gg Z_i\,. \tag{45}$$

In other words, the characteristic impedances of the output media (the chest) and the outer layer $j = 2$ are significantly greater than the characteristic impedance of the input media (air), while the inner layer $j = 1$ must be made of something with impedance no less than that of the air (that is because we really do not have a practical material with a lower impedance than air.)

In the body of the paper, we assume that $\alpha < 0$ and $\beta > 0$. By inspection of Eq. (43) with the assumptions of Eqs. (45), we see that $\alpha < 0$ is *guaranteed* if $Z_2 < Z_0$. In other words, when the garment is "soft."

While the condition $Z_2 < Z_0$ is sufficient to ensure that $\alpha < 0$, it is not necessary, and we find that $\alpha < 0$ for many choices of $Z_2 > Z_0$.

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