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A METHOD TO CUT CONVEX POLYHEDRONS AND ITS APPLICATIONS TO ILL-POSED PROBLEMS

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We first consider linear ill-posed problems on compact sets of a special structure. Second, two approaches based on the method to cut convex polyhedrons for estimation of an error of an approximate solution are proposed. Finally, the domain the exact solution of the inverse problem for the heat conduction equation belongs to is constructed.

1. Problem statement. The main feature of ill-posed problems is the impossibility to estimate the proximity of an approximate solution of a problem to its exact one [1, 2]. However, if it is known that the exact solution belongs to some compact set, then the problem becomes well-posed and an error estimation of the approximate solution is possible. In this paper, we consider two approaches to error estimation under the condition that the solution belongs to a compact set of a special structure.

Many linear problems may be reduced to solving the operator equation

$$B\bar{\varphi} = \bar{\psi}, \qquad \bar{\varphi} \in \Phi, \qquad \bar{\psi} \in \Psi \tag{1}$$

where B is a linear continuous operator acting from a linear normed space Φ into a linear normed space Ψ. In practice, exact forms of B and $\bar{\psi}$ are often unknown. Instead, we have an approximate linear continuous operator B_h and an approximate right-hand side ψ_δ such that $\forall \varphi \in \Phi$: $\|B\varphi - B_h\varphi\|_\Psi \leqslant h\|\varphi\|_\Phi$, $\|\bar{\psi} - \psi_\delta\|_\Psi \leqslant \delta$, where $h \geqslant 0$ is an error of the approximate operator and $\delta \geqslant 0$ is an error of the approximate right-hand side.

2. Approximate solution. Assume that the exact solution $\bar{\varphi}$ of problem (1) belongs to a compact set M and the operator B performs the one-to-one mapping of M onto $BM \subset \Psi$. As is shown [2], the set

$$\Phi_M \equiv \{\varphi \in M : \|B_h\varphi - \psi_\delta\|_\Psi \leqslant h\|\varphi\|_\Phi + \delta\}$$

may be adopted as a set of approximate solutions to problem (1). Denoting $\eta = (h, \delta)$, we can write $\varphi_\eta \to \bar{\varphi}$ in Φ as $\eta \to 0$.

In order to find an approximate solution φ_η of problem (1), it is convenient to use finite-dimensional Euclidean spaces. One can take an n-dimensional Euclidean space Z^n and a k-dimensional Euclidean space U^k such that approximate elements of the spaces Φ and Ψ can be written as linear functions of vectors $z \in Z^n$ and $u \in U^k$, respectively. Thus, the operator B_h is transformed into an operator A (i.e., into an $n \times k$ matrix) and the approximate right-hand side ψ_δ of the operator equation (1) is transformed into a vector $u_\Delta \in U^k$. Then, the problem of finding an approximate solution to (1) is reduced to that of finding of an element

$$z_\eta \in \{z \in Z_M \subset Z^n : \|Az - u_\Delta\| \leqslant \Delta(B_h, \psi_\delta, h, \delta, M)\} \tag{2}$$

3. The first approach to error estimation. Since $\bar{\varphi}$ belongs to the compact set M, there exists a set Z_M of a priori restrictions for a vector z in Z^n. We suppose that Z_M is a convex set in Z^n. It is shown in [2] that when a piecewise linear approximation of convex or monotone functions bounded above and below by certain constants is used, the vector z of grid values of given functions forms the set Z_M of convex polyhedrons.

Let us introduce the set

$$Z^\Delta \equiv \{z \in Z^n : \|Az - u_\Delta\| \leqslant \Delta\}$$

As an approximate solution of problem (2), then, we take any element $z_\eta \in Z_M^\Delta \equiv Z^\Delta \cap Z_M$ if $Z_M^\Delta \neq \varnothing$. The set Z^Δ is an ellipsoid in Z^n. The set Z_M^Δ is convex, since it is an intersection of two convex sets [3]. We should find Z_M^Δ or a set close to this set in some sense. As this set, we can take a polyhedron W circumscribed near the set Z_M^Δ. Let us construct this polyhedron W.

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For this purpose, we first find a point $z_\eta \in Z_M^\Delta$ using, for instance, the algorithms described in [2]. Suppose that this point $z_\eta \in Z_M^\Delta$ exists, otherwise problem (1) has no solution on the set M with the error η given above.

Since the exact solution $\bar{\varphi}$ belongs to the set M, the vector z is bounded. Therefore we assume that it is possible to construct a polyhedron V such that $Z_M^\Delta \subset V$. Let $W_0 \equiv V$.

Now we consider an algorithm to construct the polyhedron W. Let the polyhedrons $W_q \subset W_{q-1} \subset \ldots \subset W_0$, $q \geqslant 0$ be constructed. The polyhedron W_{q+1} is constructed as follows. A top of the polyhedron W_q and the point z_η are connected by the segment. The plane tangent to the surface of Z_M^Δ is constructed through the point of intersection with this segment. As a result, two polyhedrons are obtained. Out of them we choose the one that contains the point z_η (if the point z_η belongs to both the polyhedrons, then we choose the one that does not contain the above-selected top of W_q). All these constructions are possible and unique by virtue of convexity of the domains being considered.

Since the polyhedron W should be close to the set Z_M^Δ, $\lim\limits_{q\to\infty} W_q = Z_M^\Delta$. Therefore, it is necessary to choose tops of the polyhedron W_q properly in order to ensure the convergence of the algorithm. For this purpose, we shall proceed as follows. At a step Q, we remember all tops of the polyhedron and at each subsequent step $q > Q$ we consider those tops that belong to the polyhedron W_q. Once all these tops has been identified, we remember the tops of the polyhedron obtained and reiterate this procedure.

4. A method to cut convex polyhedrons. Let us consider a numerical algorithm for constructing the polyhedron W_{q+1} from the polyhedron W_q by intersection with a half-space specified by a plane tangent to Z_M^Δ. This algorithm is called the method to cut convex polyhedrons (MCCP) and is based on the following theory.

A convex polyhedron can be considered as an intersection of half-spaces bounded by planes.

Definition 1. A face of a convex polyhedron is called the intersection of this polyhedron with one of the constituent planes.

Definition 2. Let x_1 and x_2 be tops of a convex polyhedron. The segment x_1x_2 is called the edge of this polyhedron if any interior point x of the segment x_1x_2 is a boundary point for all the faces containing this point.

It is important in this definition that the point x is an interior point of the segment, since if the point x coincides with one of the tops (x_1 or x_2), then such a definition does not correspond to intuitive notions about an edge. This statement is easy to verify in the two-dimensional case.

Lemma. *A segment x_1x_2 connecting arbitrary tops x_1 and x_2 of a convex polyhedron W is an edge of this polyhedron if and only if for any interior point x of the segment x_1x_2 and for any two points $\hat{x}_1, \hat{x}_2 \in W$ that do not belong to the segment x_1x_2 the point x lies off the segment connecting the points $\hat{x}_1$ and $\hat{x}_2$.*

Proof. Let the segment x_1x_2 be the edge of the polyhedron W and x be an interior point of the segment x_1x_2. Consider arbitrary points $\hat{x}_1$ and $\hat{x}_2$ such that the point x belongs to the segment $\hat{x}_1\hat{x}_2$. Let the point $\hat{x}_1$ belong to the polyhedron W and do not lie on the edge x_1x_2. The edge x_1x_2 is formed by the intersection of several faces. The point x belongs to the edge x_1x_2. Therefore, the line that passes trough the points $\hat{x}_1$ and $\hat{x}_2$ intersects at least one of these faces, since this line does not contain the edge x_1x_2 of the polyhedron W. This implies that the point $\hat{x}_2$ lies outside the polyhedron W. In this case, for any two points $\hat{x}_1, \hat{x}_2 \in W$ that do not belong to the segment x_1x_2 the point x lies off the segment $\hat{x}_1\hat{x}_2$. The direct proposition is proved.

Now we prove the converse proposition. Let us consider arbitrary points $\hat{x}_1, \hat{x}_2 \in W$ that do not lie on the segment x_1x_2. Let an interior point x of the segment x_1x_2 do not belong to the segment $\hat{x}_1\hat{x}_2$. We assume that the convex polyhedron W is considered in the n-dimensional space. Let us construct an ε-neighborhood of the point x. Consider the segment of the line connecting points x_1 and x_2 whose endpoints are at the distance ε from the point x. Interior points of this segment are the points of the polyhedron W lying in the ε-neighborhood. Let us construct a point P that lies in the ε-neighborhood and does not belong to the polyhedron W. From the ε-neighborhood we consider an arbitrary point P^* that belongs to the ε-neighborhood and does not belong to the segment x_1x_2. If the point P^* does not belong to the polyhedron W, then the point P is constructed ($P = P^*$). Otherwise, we draw a line through the points P^* and x and find a segment on this line whose interior points lie in the ε-neighborhood of the point x. This segment is divided into two equal segments by the point x. As a point P, we can take any interior point of one of these segments that does not contain the point P^*. Assume that the point P belongs to the polyhedron W. Then if we set $\hat{x}_1 = P^*$ and $\hat{x}_2 = P$, we come to the conclusion that $\hat{x}_1, \hat{x}_2 \in W$ do not lie on the segment x_1x_2, although the point x belongs to the segment $\hat{x}_1\hat{x}_2$. This is a contradiction. Therefore, any ε-neighborhood of the point x contains points such that some of them belong to the polyhedron W and the others do not. This means that the point x is a boundary point of the set W. The boundary of the convex polyhedron W is formed by the intersection of faces, and the point x belongs to some of them. A face presents a convex polyhedron in the $(n-1)$-dimensional space, since it is formed by the intersection of a plane and the convex polyhedron W. We repeat the same reasonings as for the n-dimensional space. Thus, the point x is a boundary point for the face being considered and, hence, for all faces containing the point x. Therefore, the segment x_1x_2 is an edge. The lemma is proved.

Theorem. *Tops x_1 and x_2 of a convex polyhedron W are connected by the edge of this polyhedron if and only if for any top of the polyhedron W not coinciding with the tops x_1 and x_2 the set of faces that pass through the given top* *does not contain all faces in common for the points x_1 and x_2.*

Proof. Let the tops x_1 and x_2 of the polyhedron W be connected by an edge. By x_3 we denote another top of the polyhedron W. The faces in common for the points x_1 and x_2 are intersected along the edge x_1x_2. Consider any segment with the endpoint x_3 intersecting the segment x_1x_2 at an interior point x. This segment intersects at least one of these faces. Otherwise, there are points $\hat{x}_1 = x_3, \hat{x}_2 \in W$ such that they do not lie on the line x_1x_2 and $x \in \hat{x}_1\hat{x}_2$. According to the lemma, the segment x_1x_2 is not an edge. Therefore, the set of faces that pass through the top x_3 does not contain at least one face in common for the points x_1 and x_2.

Now we prove the converse proposition. Let a point x belong to the segment x_1x_2. Consider a face the point x belongs to. The point x can be either a boundary point or an interior point for the face. By x_3 we denote any polyhedron's top belonging to the face and not coinciding with the tops x_1 and x_2. We construct the ray x_3x. Consider any point of the ray that does not lie on the segment x_3x. This point does not belong to the polyhedron W, since the ray x_3x intersects at least one face bounding the polyhedron W. Therefore, x is a boundary point for all polyhedron's faces containing the given point. In that case, the segment x_1x_2 is an edge of the convex polyhedron. The theorem is proved.

Let us discuss the MCCP. The information on coordinates of tops of the polyhedron W_q is enough to construct the polyhedron W_{q+1}. However, it is convenient to define the polyhedron W_q by the following data: coordinates of its tops; numbers of faces an arbitrary top belongs to; numbers of tops an arbitrary top is connected by edges.

The polyhedron W_{q+1} is formed by the intersection of W_q and the half-space being considered.

Definition 3. A top of the polyhedron W_q is a cut point if this top lies outside the half-space.

Definition 4. A top of the polyhedron W_q is a boundary point if this top belongs to the plane forming the polyhedron W_{q+1}.

Definition 5. A top of the polyhedron W_q is an interior point if this top lies inside the half-space.

Definition 6. A point of the polyhedron W_q is a new point if this top is formed by the intersection of the edge connecting a pair "interior point–cut point" and the plane forming the polyhedron W_{q+1}.

If all tops of the polyhedron W_q are cut or boundary points that do not belong to a single face, then the polyhedron W_{q+1} is an empty set.

We number all new points. If an interior point forms a pair "interior point–cut point", then the edge connects it with the new point that lies on the appropriate segment. For the interior point, therefore, we replace the number of the corresponding cut point by the number of the new point. Then, the number of tops the interior point is connected by the edges of the polyhedron W_q is the same for the polyhedron W_{q+1}.

Assume that all polyhedrons lie in the n-dimensional space. We note several simple propositions:

1. If two boundary points of the polyhedron W_q are connected by an edge, they are also connected by this edge in the polyhedron W_{q+1}.

2. Let the sum of the number of tops connected with a given top by edges and the number of tops that have not been checked yet be equal to n. Then all non-checked tops are connected by edges with the given top.

3. If for a given pair of tops the number of common faces is less than $(n-1)$, then this pair is not connected by an edge.

The last proposition is obvious, since a straight line is determined by a one-parameter equation.

Remark. For the two- and three-dimensional spaces the third proposition is a necessary and sufficient criterion to select pairs of tops: a pair of tops is connected by an edge if and only if the number of common faces is equal to $(n-1)$. For spaces with a greater dimension ($n \geqslant 4$) the last statement is not valid.

Example. Let a unit four-dimensional cube whose top is at $(0,0,0,0)$ and whose edges are parallel to the coordinate axes be intersected with the half-space $x_1 - 2x_2 - 2x_3 \leqslant 0$. Then, the points $(0,0,0,0)$ and $(0,0,0,1)$ are connected by an edge and belong to the four common faces: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_1 - 2x_2 - 2x_3 = 0$.

We consider new and boundary points and find the points with which they are connected by edges. Boundary points that are connected with the boundary or interior points by edges are also connected with the appropriate tops of W_{q+1}. Each new point is connected with one interior point from the appropriate pair "interior point–cut point" by an edge. Let us find pairs of connected points among the boundary and new points. For each pair, we find the number of common faces and their numbers (the new plane is not considered). If the number is less than $(n-2)$, the pair is not connected by an edge in the polyhedron W_{q+1}. Otherwise, we consider all remaining boundary and new points and find whether there is a top that also belongs to these common faces. If such a top does not exist, the pair of tops is connected by an edge.

After the cutting of the polyhedron W_q, a part of tops (cut points) and a part of the edges may not belong to the polyhedron W_{q+1}. Therefore, it is necessary to enumerate all the tops and planes. After that, it is possible to pass on to the next cut.

The method works well for problems of a small dimension. For problems of a large dimension we should work with large arrays. This essentially reduces the computation speed. Therefore, it is better to use the method being described below.

5. The second approach to error estimation. When solving ill-posed problems, it is usually necessary to find not the set Z_M^Δ but a maximal value of the linear function $f(z) = \sum_{i=1}^{n} c_i z_i = (c, z)$. Therefore, we propose a method for solving linear programming problems on convex sets (these sets are intersections of a convex polyhedron and a some convex body with a smooth surface; for many ill-posed problems, this body is an ellipsoid).

Let at some step the point z^p be constructed. We want to construct a point z^{p+1}. For this purpose, we draw the ray with the directing vector c through z^p and find the point z^* of its intersection $\tilde{z} \neq z^*$ with the surface S of the convex set Z_M^Δ. We consider the point $\breve{z} = \frac{1}{2}(z^* + \tilde{z})$. If $n > 2$, then it is necessary to construct the line that passes through $\breve{z}$ and whose directing vector is orthogonal to the ray r and to the vector c. We find cross points of the given line with S and take their half-sum as a new point $\breve{z}$. Similarly, we build a new line orthogonal to these three vectors and repeat this procedure $(n-2)$ times. The procedure accelerates the convergence of the algorithm essentially. We suppose $z^{p+1} = \breve{z}$. For the sequence $\{z^p\}$ the following is valid: $\lim\limits_{p\to\infty} f(z^p) = \max\limits_{z\in Z_M^\Delta} f(z)$.

In the numerical realization of the method, it is necessary to use MCCP because of the following two reasons. The first reason consists in the fact that for some problems the point z^* may be close or may belong to a domain of intersection of several faces. Then, in an unsuccessful choice of the vectors r, the points z^p will not maximize the function $f(z)$ fast enough. This will lead to an essential deceleration of the computation speed or to the program termination when a value of the function $f(z)$ may be considerably different from its maximal one. The second reason consists in the desire to realize the criterion for the program to stop when the value $f(z^p)$ is close to the maximum.

Therefore, in order to construct the vector r, it is necessary to construct a polyhedron X containing the point z^* inside it. Then, one should consider the intersection of the given polyhedron with all half-spaces whose boundary planes are at a certain distance $R > 0$ from the point z^*. We may take a rectangular pyramid as an initial polyhedron X, because this pyramid has $(n+1)$ tops. Such a small number of tops is very important at large n. It is necessary to use an optimal value of the parameter R, in order to easily construct a long vector r in the polyhedron $\widetilde{X}$ and, at the same time, to use as smaller number of planes as possible. For the construction of the vector r, only very close faces of the polyhedron Z_M are taken into account. Therefore, it is clear that if one chooses R very small, then, in the presence of other faces close to the point z^*, the point $\tilde{z}$ constructed using the vector r will be close to the point z^*. Clearly, the number of iterations for the determination of a maximum of $f(z)$ increases. If the value R is large, we should work with large arrays and the computation time increases, too.

It may happen that at some stage all tops of the polyhedron $\widetilde{X}$ lie at the distance $R_0 < R$ from the point z^*. In that case, we should assume that a point maximizing the function $f(z)$ is found. Thus, with the help of MCCP, it is possible to formulate a criterion for the program to stop.

The method to construct the vector r and the criterion for the program to stop should be supplemented by the following. The problem consists in the maximization of the function $f(z)$ on the set $Z_M^\Delta = Z_M \cap Z^\Delta$ but not on Z_M. However, while constructing the polyhedron $\widetilde{X}$ around the point z^*, only faces of the polyhedron Z_M are taken into account and the presence of the smooth surface of the set Z^Δ is not taken into account. Therefore, it is necessary to do the following. One should construct the polyhedron being considered only with regard to the boundary of the set Z_M. If the polyhedron obtained has tops lying outside the set Z^Δ, one should construct a polyhedron using these tops and the point z^* as was described above in the method for the construction of the convex polyhedron W circumscribed around Z_M^Δ.

6. The inverse problem for the heat conduction equation. As an example, we consider the following inverse problem for the heat-conduction equation:

$$\begin{cases} u_t = u_{xx}, \\ u(0,t) = 0, \\ u(\pi,t) = 0, \end{cases} \quad \begin{matrix} u(x,0) = \varphi(x), \\ u(x,T) = \psi(x), \end{matrix} \quad \varphi(x),\ \psi(x) \in L_2[0,\pi]$$

The function $\psi(x)$ is given. We want to find the function $\varphi(x)$ on the set of functions convex up on the segment $[0,\pi]$ and restricted from above by a constant $C > 0$.

Let us use the method of separation of variables and the expansions of functions $\varphi(x)$ and $\psi(x)$ in sine series. We write down:

$$\varphi(x) = \frac{2}{\pi}\sum_{l=1}^{\infty} S_l \sin(lx), \quad \psi(x) = \frac{2}{\pi}\sum_{r=1}^{\infty} U_r \sin(rx), \quad S_l = \int_0^\pi \varphi(x)\sin(lx)\,dx, \quad U_r = \int_0^\pi \psi(x)\sin(rx)\,dx$$

Since $U_l = S_l e^{-l^2 T}$, we may use an $(n\times k)$ matrix A that transforms the vector z of grid values $\{z_i\}_1^n$ of the function $\varphi(x)$ given on the grid $\{x_i\}_1^n$ into the vector of first k Fourier coefficients of the function $\psi(x)$:

$$A_{li} = \frac{1}{l^2}\left(\frac{\sin(lx_i)-\sin(lx_{i+1})}{x_{i+1}-x_i} + \frac{\sin(lx_i)-\sin(lx_{i-1})}{x_i - x_{i-1}}\right) e^{-l^2 T}, \quad l=\overline{1,k}, \quad i = \overline{1,n}$$

Thus, we pass on from functions $\varphi(x)$ and $\psi(x)$ square integrable on the segment $[0,\pi]$ to the vector of grid values and to the vector of first k Fourier coefficients, respectively. The transformation from the function $\psi(x)$ to the vector of grid values is used in many monographs. Instead, the transformation to the vector of first k Fourier coefficients allows us to find an error δ. As the initial information, we use grid values $\{y_j\}_1^m$ of the function $\psi_\delta(x)$ on the grid $\{\hat{x}_j\}_1^m$ and a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ such that

$$|y_j - \psi_\delta(\hat{x}_j)| \leqslant \xi_j, \quad j = \overline{1,m}$$

We suppose that the function $\bar{\psi}(x)$ can be written as the finite Fourier series: $\bar{\psi}(x) = \dfrac{2}{\pi}\sum\limits_{l=1}^{k} V_l \sin(lx)$. The function $\varphi(x)$ is bounded on the segment $[0,\pi]$. The supposition does not reduce the generality, since grid values of any function $\psi_\delta(x)$ can be considered as grid values of the function $\psi_\delta(x)$ representable as a finite Fourier series but with a changed vector $\breve{\xi}$ of errors. Write down a new vector of errors:

$$\breve{\xi}_j = \xi_j + \frac{2}{\pi}\sum_{l=k+1}^{\infty} |S_l|_{\max} e^{-l^2 T} |\sin(lx_j)|, \quad j = \overline{1,m}$$

Taking into account the grid values y_j, $j=\overline{1,m}$, and the changed vector of errors $\breve{\xi}_j$, $j=\overline{1,m}$, we write down:

$$Y_1 \leqslant Du \leqslant Y_2, \quad Y_1 \leqslant Dv \leqslant Y_2 \tag{3}$$

where

$$u = (U_1, U_2, \ldots, U_k), \quad Y_{1j} = y_j - \breve{\xi}_j, \quad Y_{2j} = y_j + \breve{\xi}_j, \quad D_{ji} = \frac{2}{\pi}\sin(ix_j), \quad j=\overline{1,m}, \quad i = \overline{1,k}$$

For the error δ we have: $\delta^2 = \dfrac{2}{\pi}\sum\limits_{l=1}^{k}(U_l - V_l)^2$. By virtue of boundedness of the function $\varphi(x)$ on the segment $[0,\pi]$, we can construct vectors $v_{\min}$ and $v_{\max}$ such that

$$v_{\min} \leqslant u \leqslant v_{\max}, \quad v_{\min} \leqslant v \leqslant v_{\max} \tag{4}$$

Conditions (3) and (4) allow us to find the error δ using the grid values $\{y_j\}_1^m$ and the vector ξ of errors. For this purpose, using MCCP, we construct the domain M_V the vectors u and v belong to. We take a vector u such that

$$\delta \equiv \sup_{v\in M_V}\sum_{l=1}^{k}(U_l - V_l)^2 = \inf_{u^*\in U^k}\left(\sup_{v\in M_V}\sum_{l=1}^{k}(U_l^* - V_l)^2\right)$$

The problem of the determination of the error δ and the vector u is reduced to the determination of tops of the

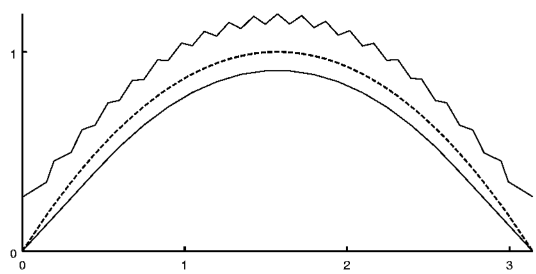

Fig. 1. The exact solution and the area this solution belongs to ($\Delta = 0.29$)

polyhedron M_V. If a crude estimate of the error is used, then the problem is reduced to the determination of minimal and maximal values of each coordinate for tops in the polyhedron M_V. In the last case, we can write

$$\delta^2 = \frac{1}{2\pi} \sum_{l=1}^{k} (V_{l\,\max} - V_{l\,\min})^2, \quad U_l = \frac{1}{2} (V_{l\,\max} + V_{l\,\min})$$

7. Example.

Let $\bar{\varphi}(x) = \dfrac{4}{\pi^2}(\pi - x)x$ be the exact solution; as a right-hand side we take $\psi_\delta(x) = B\bar{\varphi}(x)$. Let $T = 10^{-2}$, $\Delta = 0.29$, $C = 1.2$, $k = 10$, $n = 20$. The exact solution and the area this solution belongs to are represented in Figure 1. This area is constructed using the method described in Section 5 of the paper.

All the algorithms described above were implemented in Fortran 90 (Microsoft Fortran PowerStation 4.0).

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