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МЕТОД ЛОКАЛЬНОГО ПОИСКА ДЛЯ НЕВЫПУКЛОЙ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ С ФУНКЦИОНАЛОМ БОЛЬЦА

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Рассматривается невыпуклая задача оптимального управления, в которой невыпуклость порождается интегрально-терминальным целевым функционалом. Предлагается новый метод локального поиска, который позволяет получить управляемый процесс, в частности, удовлетворяющий принципу максимума Понтрягина. Исследуются некоторые особенности сходимости метода. Кроме того, проведен вычислительный эксперимент, результаты которого свидетельствуют о конкурентоспособности и эффективности алгоритма.

Ключевые слова: невыпуклые задачи оптимального управления, принцип максимума Понтрягина, метод локального поиска.

1. Introduction. During the recent two decades, specialists in the theory and methods of Optimal Control (OC) have been witnessed the considerable demand — characteristic for such applied fields as engineering, economics, space sciences, etc. — for the mathematical tools needed to solve different kinds of OC problems characterized by various types of nonconvexity generated by distinct nonconvex structures, often invisible, implicitly represented in the problem under study. Let us mention only the following two well-known cases: case one typical for nonlinear (with respect to (w.r.t.) the state) control systems and case two typical for hierarchical systems.

So, this is how the problem is stated. However, not all the scientists realize the degree of its complexity. It seems that most of mathematicians are not ready to attack the various aspects even of visible nonconvexities, which play the crucial role in finding local and (moreover) global solutions.

In this paper we study only one type of nonconvexities generated by the Bolza objective functional. On the other hand, the objective is also rather simple: to construct a special local search method allowing one to obtain a stationary (in the sense of Pontryagin's maximum principle (PMP)) point.

Note that this aim — to create a special local search method for each kind of nonconvex problems — has not been reached and fixed until present for the OC problem, while — as far as finite dimensional problems are concerned — some results have been obtained for different kinds of nonconvex problems (d.c. minimization, convex maximization, d.c. constraint problems, etc.) [7, 10]. In our opinion, only after creating special Local Search Algorithms (LSA) for various types of nonconvex OC problems, one can begin with constructing Global Search Procedures (based, say, on Global Optimality Conditions) allowing one to avoid the use of a stationary (PMP) process with improving the value of the objective functional on various nonconvex OC problems [10].

This paper is organized as follows. In Section 2 a statement of the OC problem under standard (for OC) assumptions is discussed, the focus being made on the principal nonconvexity of the problem generated by the objective functional. In Section 3 we present the solution of the so-called linearized problem in which all the nonconvexities are linearized. In Section 4 we discuss a local search method. In Section 5 we study the convergence features of the sequence $\{x^s(\cdot), u^s(\cdot)\}$ generated by the proposed algorithm. In Section 6 some results of the preliminary numerical testing of LSA are presented and commented. In Section 7 some brief remarks on the content of the paper are given.

2. Formulation of the problem. Consider the following control system:

$$\dot{x}(t) = A(t)x(t) + B\big(u(t), t\big) \quad \overset{\circ}{\forall} t \in T = [t_0, t_1], \quad x(t_0) = x_0 \in \mathbb{R}^n. \tag{1}$$

Here the matrix $A(t)$ has the elements $t \mapsto a_{ij}(t)$, i, $j = 1, 2, \ldots, n$, from $L_\infty(T)$, the mapping $(u, t) \mapsto B(u, t)$: $\mathbb{R}^{r+1} \to \mathbb{R}^n$ is continuous w.r.t. every $u \in \mathbb{R}^r$ and every $t \in \mathbb{R}$. Further, the control $u(\cdot)$ satisfies the standard assumptions of OC:

$$u(\cdot) \in \mathcal{U} = \big\{\, u(\cdot) \in L^r_\infty(T) \mid u(t) \in U \quad \overset{\circ}{\forall} t \in T \,\big\}, \tag{2}$$

where the set U is compact in $\mathbb{R}^r$. Here the sign $\overset{\circ}{\forall}$ means "for almost all".

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Under these assumptions, for each $u(\cdot) \in \mathcal{U}$ there exists a unique solution $x(t) = x(t,u)$, $t \in T$, to the original Cauchy problem (1) such that $x(\cdot) \in AC^n(T)$ (i.e., absolutely continuous) [2, 3].

The goal of this control consists in the maximization of the Bolza functional

$$(\mathcal{P}):\quad J(u) := \varphi\big(x(t_1)\big) + \int_T \Big[g\big(x(t),t\big) + f\big(u(t),t\big)\Big]\,dt \uparrow \max_u, \quad u \in \mathcal{U}, \tag{3}$$

over the control system (1)–(2), i.e., $x(t) = x(t,u)$, $t \in T$, $u \in \mathcal{U}$ in (3). Here the function $x \mapsto \varphi(x)\colon \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable on a rather large open convex set $\Omega \subset \mathbb{R}^n$ and the function $(u,t) \mapsto f(u,t) : \mathbb{R}^{r+1} \to \mathbb{R}$ is a continuous w.r.t. each variable.

The function $(x,t) \mapsto g(x,t)$ is also continuous w.r.t. each variable and, moreover, is convex and differentiable w.r.t. the first variable x on $\Omega \subset \mathbb{R}^n$. Under such assumptions, Problem $(\mathcal{P})$–(1)–(3) turns out to be nonconvex, so that it may possess a number of locally optimal and stationary (satisfying the PMP) processes $\big(x_*(\cdot), u_*(\cdot)\big)$ different from a global one $\big(z(\cdot), w(\cdot)\big)$, $z(t) = x(t,w)$, $t \in T$, $w(\cdot) \in \mathcal{U}$, even w.r.t. the values of the objective functional. Some examples of such problems can be found in [9, 10].

In the next sections of the paper, we discuss some regular procedures allowing us to obtain a stationary process for Problem $(\mathcal{P})$–(1)–(3).

3. Linearized problem. Under the assumptions of the previous section, let us consider the maximization problem

$$\big(\mathcal{PL}(y)\big):\quad I_y(u) := \Big\langle \nabla\varphi\big(y(t_1)\big), x(t_1,u)\Big\rangle + \int_T \Big[\Big\langle \nabla g\big(y(t),t\big), x(t,u)\Big\rangle + f\big(u(t),t\big)\Big]\,dt \uparrow \max_u, \quad u \in \mathcal{U}, \tag{4}$$

for the control system (1)–(2), where $y(t) \in \mathbb{R}^n$, $t \in [t_0,t_1]$, is a given continuous function. It is well-known that, in the convex Problem $\big(\mathcal{PL}(y)\big)$–(4), the PMP turns out to be a necessary and sufficient condition for the process $\big(x_*(t), u_*(t)\big)$ being (globally) optimal in $\big(\mathcal{PL}(y)\big)$. More precisely, if $\big(x_*(t), u_*(t)\big)$ is a solution to $\big(\mathcal{PL}(y)\big)$–(4), then the maximum condition

$$H_L\big(x_*(t), u_*(t), \psi(t), t\big) = \max_{v \in U} H_L\big(x_*(t), v, \psi(t), t\big) \quad \overset{\circ}{\forall} t \in T, \tag{5}$$

holds with the Pontryagin function for the problem $\big(\mathcal{PL}(y)\big)$–(4):

$$H_L(x,u,\psi,t) = \big\langle \psi, A(t)x + B(u,t)\big\rangle + \Big\langle \nabla g\big(y(t),t\big), x\Big\rangle + f(u,t). \tag{6}$$

Here the function $\psi(t) = \psi_y(t)$, $t \in [t_0,t_1]$, is a unique absolutely continuous solution to the adjoint system

$$\dot\psi(t) = -\psi(t)A(t) - \nabla g\big(y(t),t\big), \quad t \in T, \quad \psi(t_1) = \nabla\varphi\big(y(t_1)\big). \tag{7}$$

Taking into account the special form (6) of the Pontryagin function, we can rewrite the maximum condition (5) in the following form:

$$\Big\langle \psi(t), B\big(u_*(t),t\big)\Big\rangle + f\big(u_*(t),t\big) = \max_{v \in U}\Big[\big\langle \psi(t), B(v,t)\big\rangle + f(v,t)\Big] \quad \overset{\circ}{\forall} t \in T. \tag{5$'$}$$

Thus, one can easily conclude that in order to solve Problem $\big(\mathcal{PL}(y)\big)$ we have to implement the following procedure.

Step 1. Solve the adjoint system (7).

Step 2. Find the control $u_*(\cdot) \in \mathcal{U}$ according to the maximum condition (5′) with the possible application of classical optimization methods, such as quasi-Newton, SQP, etc., taking into account that at each time instant $t \in T$ we need a global solution $u_*(t)$ of problem (5′).

Step 3. Solve the system (1) of ordinary differential equations (ODEs) with the control $u_*(t)$, $t \in T$.

Stop. The process $\big(x_*(\cdot), u_*(\cdot)\big)$, $x_*(t) = x(t,u_*)$, $t \in T$, is a solution to Problem $\big(\mathcal{PL}(y)\big)$–(4).

4. A local search method. At the end of solving Problem $(\mathcal{P})$–(1)–(3), the following procedure has shown to be rather efficient for finite-dimensional problems. In the case of Optimal Control, this procedure can be described as follows.

Once a feasible control $u^s(\cdot) \in \mathcal{U}$ is given, the next iteration $u^{s+1}(\cdot) \in \mathcal{U}$ is chosen as an approximate solution to the linearized problem

$$(\mathcal{PL}_s): \quad I_s(u) := \Big\langle \nabla\varphi\big(x^s(t_1)\big), x(t_1,u)\Big\rangle + \int_T \Big[\Big\langle \nabla g\big(x^s(t),t\big), x(t,u)\Big\rangle + f\big(u(t),t\big)\Big]\, dt \uparrow \max_u, \quad u \in \mathcal{U}, \tag{8}$$

where $x(t,u)$ and $x^s(t) = x(t,u^s)$, $t \in [t_0,t_1]$, are solutions to the system (1) of ODEs with $u(\cdot)$ and $u^s(\cdot)$, respectively.

The issue of convergence of the sequence $\big\{x^s(\cdot), u^s(\cdot)\big\}$ generated by the above procedure emerges immediately.

On the other hand, it is clear from the previous section that the solution $\big(x^{s+1}(\cdot), u^{s+1}(\cdot)\big)$ to Problem $\big(\mathcal{PL}_s\big)$–(8) can be obtained by solving the adjoint system

$$\dot\psi^s(t) = -\psi^s(t)A(t) - \nabla g\big(x^s(t),t\big), \quad \psi(t_1) = \nabla\varphi\big(x^s(t_1)\big) \tag{9}$$

with consideration of the maximum condition

$$\Big\langle \psi^s(t), B\big(u^{s+1}(t),t\big)\Big\rangle + f\big(u^{s+1}(t),t\big) = \max_{v\in U}\Big[\big\langle \psi^s(t), B(v,t)\big\rangle + f(v,t)\Big] \ \ \overset{\circ}{\forall} t \in T, \tag{10}$$

which provides for the control $u^{s+1}(\cdot) \in \mathcal{U}$. After that, the state $x^{s+1}(t)$ is computed as a solution to the control system (1) corresponding to the control $u^{s+1}(\cdot) \in \mathcal{U}$.

This idea leads us to a more realistic algorithm whose principal steps have been discussed above.

Let there be given a sequence of numbers $\{\delta_s\}$ such that

$$\delta_s > 0, \quad s = 0,1,2,\ldots, \quad \sum_{s=0}^{\infty} \delta_s < +\infty, \tag{11}$$

and a current process $\big(x^s(\cdot), u^s(\cdot)\big)$, $u^s(\cdot) \in \mathcal{U}$, $x^s(t) = x(t,u^s)$, $t \in [t_0,t_1]$.

Having the state $x^s(\cdot) \in AC^n(T)$, one can solve the corresponding adjoint system (9). After that we construct a control $u^{s+1}(\cdot) \in \mathcal{U}$ by solving the finite dimensional problem approximately almost everywhere over T with consideration of (10), so that the following inequality holds:

$$\Big\langle \psi^s(t), B\big(u^{s+1}(t),t\big)\Big\rangle + f\big(u^{s+1}(t),t\big) + \frac{\delta_s}{t_1 - t_0} \geqslant \sup_{v\in U}\Big[\big\langle \psi^s(t), B(v,t)\big\rangle + f(v,t)\Big] \ \ \overset{\circ}{\forall} t \in T. \tag{12}$$

Now we are ready to study the convergence of the sequence $\big\{x^s(\cdot), u^s(\cdot)\big\}$ generated by the above procedure. First, from (12) it immediately follows that

$$\begin{aligned} 0 \leqslant W_s &:= \int_T \sup_{v\in U}\Big[\Big\langle \psi^s(t), B(v,t) - B\big(u^s(t),t\big)\Big\rangle + f(v,t) - f\big(u^s(t),t\big)\Big]\, dt \leqslant \\ &\leqslant \delta_s + \int_T \Big[\Big\langle \psi^s(t), B\big(u^{s+1}(t),t\big) - B(u^s(t),t)\Big\rangle + f\big(u^{s+1}(t),t\big) - f\big(u^s(t),t\big)\Big]\, dt. \end{aligned} \tag{13}$$

Further, it can readily be seen from systems (1) and (9) that, for each solution $x(t) = x(t,u)$, $t \in T$, to system (1) corresponding to a feasible control $u(\cdot) \in \mathcal{U}$, one has

$$\int_T \Big\langle \psi^s(t), B\big(u(t),t\big)\Big\rangle\, dt = \int_T \Big\langle \nabla g\big(x^s(t),t\big), x(t,u)\Big\rangle\, dt + \Big\langle \nabla\varphi\big(x^s(t_1)\big), x(t_1,u)\Big\rangle - \big\langle \psi^s(t_0), x_0\big\rangle.$$

From this equality it follows that estimate (13) takes the form

$$\begin{aligned} 0 \leqslant W_s \leqslant \int_T \Big[\Big\langle \nabla g\big(x^s(t),t\big), x^{s+1}(t) - x^s(t)\Big\rangle + f\big(u^{s+1}(t),t\big) - f\big(u^s(t),t\big)\Big]\, dt \, + \\ + \Big\langle \nabla\varphi\big(x^s(t_1)\big), x^{s+1}(t_1) - x^s(t_1)\Big\rangle + \delta_s = I_s\big(u^{s+1}\big) - I_s\big(u^s\big) + \delta_s; \end{aligned} \tag{14}$$

hence, with the help of convexity of $g(\cdot, t)$, $t \in T$, and $\varphi(\cdot)$ one obtains

$$
\begin{aligned}
0 \leqslant W_s \leqslant \int_T \Big[g\big(x^{s+1}(t), t\big) - g\big(x^s(t), t\big) + f\big(u^{s+1}(t), t\big) - f\big(u^s(t), t\big) \Big]\, dt\, + \\
+ \varphi\big(x^{s+1}(t_1)\big) - \varphi\big(x^s(t_1)\big) + \delta_s = J\big(u^{s+1}\big) - J\big(u^s\big) + \delta_s.
\end{aligned} \tag{15}
$$

It follows from (15) that the number sequence $\big\{J(u^s)\big\}$ turns out to be almost monotonously increasing, i.e., $J\big(u^{s+1}\big) + \delta_s \geqslant J\big(u^s\big)$. Thus, one may conclude that, by virtue of the unboundedness from above of the objective functional for Problem $(\mathcal{P})$–(1)–(3) $\mathcal{V}(\mathcal{P}) := \sup\limits_u \big\{ J(u) \mid u \in \mathcal{U} \big\} < +\infty$ (which follows from the assumptions of Section 2; in particular, the compactness of $U \subset \mathbb{R}^r$ and the properties of the control system (1) (see [2, 3])), the sequence $\big\{J(u^s)\big\}$ converges:

$$
\exists \lim_{s\to\infty} J(u^s) = J_* \leqslant \mathcal{V}(\mathcal{P}). \tag{16}
$$

Hence, from (11) and (15) we obtain

$$
\lim_{s\to\infty} W_s = \lim_{s\to\infty} \int_T \sup_{v\in U} \Big[\Big\langle \psi^s(t), B(v,t) - B\big(u^s(t), t\big) \Big\rangle + f(v,t) - f\big(u^s(t), t\big) \Big]\, dt = 0. \tag{17}
$$

Taking into account the fact that $u^s(t) \in U \quad \overset{\circ}{\forall} t \in T$, we conclude that the expression under the integral is nonnegative almost everywhere over T; therefore, from (17) one has

$$
\lim_{s\to\infty} \sup_{v\in U} \Big[\Big\langle \psi^s(t), B(v,t) - B\big(u^s(t), t\big) \Big\rangle + f(v,t) - f\big(u^s(t), t\big) \Big] = 0 \quad \overset{\circ}{\forall} t \in T. \tag{18}
$$

On the other hand, if we consider the Pontryagin function for Problem $(\mathcal{P})$–(1)–(3) (see (6))

$$
H(x, u, \psi, t) = \big\langle \psi, A(t)x + B(u,t) \big\rangle + g(x,t) + f(u,t),
$$

then it can readily be seen that [1–3]

$$
\begin{aligned}
\Big\langle \psi^s(t), B(v,t) - B\big(u^s(t), t\big) \Big\rangle + f(v,t) - f\big(u^s(t), t\big) = H\big(x^s(t), v, \psi^s(t), t\big) - H\big(x^s(t), u^s(t), \psi^s(t), t\big) \triangleq \\
\triangleq \Delta_v H\big(x^s(t), u^s(t), \psi^s(t), t\big) =: \Delta_v H_s[t].
\end{aligned} \tag{19}
$$

Hence, it is easy to see that (18) may be rewritten in an equivalent form as follows:

$$
\lim_{s\to\infty} \sup_{v\in U} \Delta_v H_s[t] = 0. \tag{18$'$}
$$

It is well-known [1] that the PMP standard form for Problem $(\mathcal{P})$–(1)–(3) consists of the maximum condition $\sup\limits_{v\in U} \Delta_v H\big(x_*(t), u_*(t), \psi_*(t), t\big) = 0$, where $x_*(t) = x(t, u_*)$, $t \in T$, $u_*(\cdot) \in \mathcal{U}$, and the vector-function $\psi_*(t) \in \mathbb{R}^n$ is a unique absolutely continuous solution to the adjoint system

$$
\dot{\psi}_*(t) = -\psi_*(t)A(t) - \nabla g\big(x_*(t), t\big), \quad \dot{\psi}_*(t_1) = \nabla \varphi\big(x_*(t_1)\big).
$$

Based on (19) and (18$'$), we come to the following result.

Theorem 1. *The sequence of controlled processes* $\big\{x^s(\cdot), u^s(\cdot)\big\}$ *generated by rules* (9), (11), *and* (12) *satisfies the PMP in the sense of condition* (18$'$).

Thus, the sequence $\big\{x^s(\cdot), u^s(\cdot)\big\}$ satisfies the maximum condition (18$'$). Moreover, the corresponding sequence $\big\{J\big(u^s\big)\big\}$ of numbers turns out to be convergent in the usual sense (see (16)). So, due to (15) one can use the following two inequalities as the stopping criterion:

$$
J\big(u^{s+1}\big) - J\big(u^s\big) \leqslant \frac{\tau}{2}, \quad \delta_s \leqslant \frac{\tau}{2}. \tag{20}
$$

Hence, from (13) and (15) we obtain

$$
0 \leqslant W_s \triangleq \int_T \sup_{v\in U} \Big[\Big\langle \psi^s(t), B(v,t) - B\big(u^s(t), t\big) \Big\rangle + f(v,t) - f\big(u^s(t), t\big) \Big]\, dt = \int_T \sup_{v\in U} \Delta_v H_s[t]\, dt \leqslant \tau. \tag{21}
$$

Definition. The control $u^s(\cdot) \in \mathcal{U}$ satisfying (21) is said to be τ-critical.

Note that inequality (21) does not imply that

$$\sup_{v \in U} \Delta_v H_s[t] \leqslant \frac{\tau}{\text{mes } T} \quad \overset{\circ}{\forall} t \in T, \tag{22}$$

while (21) follows from (22). In addition, as follows from (14) and (15), the number sequence $\left\{I_s\big(u^s\big)\right\}$ (see (8)) is also convergent. Hence, one can use the following two inequalities as the stopping criterion:

$$I_s\big(u^{s+1}\big) - I_s\big(u^s\big) \leqslant \frac{\tau}{2}, \quad \delta_s \leqslant \frac{\tau}{2}. \tag{23}$$

In this case, as above, the control $u^s(\cdot)$ again turns out to be τ-critical. However, if we turn back to (13), then it can readily be understood that it is mostly feasible and convenient to use the following two inequalities as the stopping criterion:

$$\begin{aligned} \delta_s \leqslant \frac{\tau}{2}, \quad V_s := &\int_T \Big[H\big(x^s(t), u^{s+1}(t), \psi^s(t), t\big) - H\big(x^s(t), u^s(t), \psi^s(t), t\big)\Big]\, dt = \\ &= \int_T \Big[\Big\langle \psi^s(t), B\big(u^{s+1}(t), t\big) - B\big(u^s(t), t\big)\Big\rangle + f\big(u^{s+1}(t), t\big) - f\big(u^s(t), t\big)\Big]\, dt \leqslant \frac{\tau}{2}. \end{aligned} \tag{24}$$

Similar to the two cases considered above (see (20) and (23)), it can readily be seen that the control $u^s(\cdot) \in \mathcal{U}$ remains τ-critical. The ease of applying criterion (24) follows from the principal rule (12) of the approximate ("maximum principle") construction of the control sequence $\big\{u^s(\cdot)\big\}$, which operates at each iteration. In particular, this means that there is no need to compute the supplementary quantities $J\big(u^s\big)$ or $I_s\big(u^s\big)$, etc.

5. Additional convergence features. Below we are going to continue the study of properties of the sequence $\big\{x^s(\cdot), u^s(\cdot)\big\}$ generated by the local search procedure (LSP) (9), (11), (12). In particular, we would like to identify the conditions when the convergence of the states $\big\{x^s(\cdot)\big\}$ takes place. Suppose that the functions $x \mapsto \varphi(x)\colon \mathbb{R}^n \to \mathbb{R}$ and $x \mapsto g(x,t)\colon \mathbb{R}^n \to \mathbb{R}$ are not merely convex, but are strongly convex [3], so that the following inequalities hold for all $x, y \in \mathbb{R}^n$:

$$\begin{aligned} \varphi(x) - \varphi(y) \quad &\geqslant \big\langle \nabla\varphi(y), x - y\big\rangle + \frac{\mu}{2}\,\|x - y\|^2, \\ g(x,t) - g(y,t) &\geqslant \big\langle \nabla g(y,t), x - y\big\rangle + \frac{\chi}{2}\,\|x - y\|^2, \quad t \in T. \end{aligned} \tag{25}$$

Therefore, from (14) we obtain

$$0 \leqslant W_s \leqslant \delta_s + J\big(u^{s+1}\big) - J\big(u^s\big) - \frac{\mu}{2}\,\big\|x^{s+1}(t_1) - x^s(t_1)\big\|^2 - \frac{\chi}{2}\int_T \big\|x^{s+1}(t) - x^s(t)\big\|^2\, dt,$$

or equivalently $\displaystyle \frac{\mu}{2}\,\big\|x^{s+1}(t_1) - x^s(t_1)\big\|^2 + \frac{\chi}{2}\int_T \big\|x^{s+1}(t) - x^s(t)\big\|^2\, dt \leqslant J\big(u^{s+1}\big) - J\big(u^s\big) + \delta_s.$

Taking into account the convergence of the two number sequences $\{\delta_s\}$ and $\left\{J\big(u^s\big)\right\}$ (see (11) an (16)), we can conclude that the sequence $\big\{x^s(\cdot)\big\}$ converges in the following sense:

$$x^s(t_1) \to x_1 \quad \text{in} \quad \mathbb{R}^n, \tag{26}$$

$$x^s(\cdot) \to x_*(\cdot) \quad \text{in} \quad L_2(T). \tag{27}$$

Here the existence of the vector $x_1 \in \mathbb{R}^n$ and the function $x_*(t) \in L_2(T)$ follows from the completeness of the spaces $\mathbb{R}^n$ and $L_2(T)$ in the corresponding norms.

Theorem 2. *Suppose the assumptions of Section 2 are satisfied and the functions $\varphi(\cdot)$ and $g(\cdot,t)$, $t \in T$, are strongly convex, so that (25) holds. Then, the sequence $\big\{x^s(\cdot)\big\}$ converges in the sense of conditions (26) and (27).*

6. Numerical experiments. The LSA testing presented below has been conducted on a series of examples of dimension $(n \times r)$ from 2×2 to 20×20. This series has been constructed by means of the procedure whose idea belongs to Calamai et al. [8] and can be described as follows (see also [12]).

First, we construct several (say, four) little kernel problems of dimension, say, 1×1, 2×1 or 2×2, for which the global solutions, the local solutions, and the stationary processes can be computed analytically.

Knowing these stationary processes, we construct a "big" problem (of dimension, say 20×20) by combining the kernels and by calculating the values of the objective functional of the "big" problem for all local solutions and stationary (PMP) processes, as well as for the known global solutions. In addition, one has to mutate all the data of the constructed "big" problem by means of the matrix $H(y) := I - 2\,\dfrac{yy^T}{\langle y, y\rangle}$ with a random vector y from the interval $[-10, 10]$ (see [12]).

About 50 different nonconvex OC problems have been generated by this procedure [12]. Below one can see the first results of preliminary testing that show rather competitive features of the LSA developed. The LSA has been implemented on C++ and all computations have been executed by Intel Core 2 Duo 2.0 GHz, the on-line memory being 2 Gb.

The inequality $I_s\big(u^{s+1}\big) - I_s\big(u^s\big) \leqslant \dfrac{\tau}{2}$ (see (8)) is used as the stopping criterion, where $\tau = 10^{-3}$ and $\delta = 10^{-4}$ on all the iterations.

The following notation is used in the table below: № is the test problem number; m_i is the number of kernel problems of type i $(i = 1, 2, 3, 4)$ forming the "big" problem numbered by №; n and r are the dimensions of the problem w.r.t. the state and the control, respectively; PMP is the number of the processes satisfying the PMP in the "big" problem; $\mathcal{V}(\mathcal{P})$ is the optimal value of the problem; J_0 is the initial value of the objective functional; $J_{\rm st}$ is the value of the objective functional at the final iteration; PL is the number of the solved linearized problems; Time is the CPU running time for the LSA (sec.).

The results of LSA numerical testing

№	m_1	m_2	m_3	m_4	n	r	PMP	$\mathcal{V}(\mathcal{P})$	J_0	$J_{\rm st}$	PL	Time
1	0	0	1	1	4	3	6	45.01	10.91	23.56	11	1.12
2	1	0	0	1	4	4	9	47.97	9.66	18.23	14	1.36
3	2	1	0	0	6	6	27	56.17	13.33	22.71	17	1.64
4	0	2	0	1	6	6	27	57.96	20.07	31.16	16	1.51
5	1	0	2	0	6	4	12	55.49	12.25	24.48	18	1.87
6	3	1	1	0	10	9	162	94.15	21.08	41.62	21	2.53
7	2	1	2	0	10	8	108	91.19	22.33	37.19	27	3.19
8	1	1	1	2	10	9	162	108.21	27.40	52.61	25	2.89
9	3	2	2	0	14	12	972	126.89	32.41	67.11	34	3.72
10	3	1	1	2	14	13	1458	149.15	33.90	54.83	29	3.48
11	2	2	2	1	14	12	972	133.92	35.57	48.95	33	3.61
12	3	2	4	1	20	16	11664	189.41	47.82	71.33	45	7.43
13	3	3	0	4	20	20	59049	217.10	55.88	83.15	37	6.86
14	7	1	1	1	20	19	39366	203.53	40.49	56.43	41	7.18

As far as comments on computational simulations are concerned, we repeat again that the initial controls have been chosen in order to create the worse conditions for the operation of the LSA.

Nevertheless, in several LSA problems we has found a globally optimal process, since this is also a problem to find a convenient initial control. However, for most of test problems the global solution has not been reached.

On the other hand, we have to pay our attention to the fact that the difference between the initial value J_0 of the objective functional and the obtained value turns out to be rather considerable. So, the LSA has shown to be rather efficient on the considered series of examples. Note that for each problem the solution time is rather moderate and, it seems, it shall allow us to perform a global search for the test problems under consideration.

7. Conclusion. In this paper a nonconvex OC problem has been considered for the case when the nonconvexity has been generated by maximizing the objective functional with a convex terminal part and with an integral part having a convex (with respect to the state) integrand.

On the whole, the problem turns out to be nonconvex in the sense that there may be local solutions which are rather far from a globally optimal process even with respect to the values of the objective functional.

Further, for this problem we have proposed and substantiated the local search method based, on the one hand, on the classical linearization idea and, on the other hand, on the method of solving linearized problems described in Section 3.

In addition, the convergence of the developed algorithm has been studied. Finally, the first numerical testing of the developed algorithm has shown to be rather efficient and has demonstrated the possibility of applying the algorithm to the global search procedure that we intend to describe in our forthcoming papers.

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