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CONDITIONS OF SOURCEWISE REPRESENTATION AND RATES OF CONVERGENCE OF METHODS FOR SOLVING ILL-POSED OPERATOR EQUATIONS. PART I

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We outline some recent results on rates of convergence of regularization methods for linear ill-posed operator equations in Hilbert and Banach spaces. Special attention is paid to the necessity of sourcewise representation conditions for power estimates of convergence rates of the methods under consideration.

1. Introduction. The aim of this paper is to introduce experts in numerical methods to a certain group of recent results concerning stable methods for solving ill-posed operator equations. Since these results are of theoretical and practical interest, their systematization and detailed presentation seem to be useful.

Let $F : X_1 \to X_2$ be a nonlinear operator and X_1 and X_2 be complex Banach spaces. Our interest is focussed on equations of the form

$$F(x) = 0, \quad x \in X_1 \tag{1.1}$$

Suppose equation (1.1) possesses a solution x^* that may be not unique. Let the operator $F(x)$ be twice Gâteaux differentiable in a neighborhood of the solution x^*. We do not introduce any assumptions on existence of a continuous inverse of the linear operator $F'(x)$ in a neighborhood of x^*. Equations of this type often arise in the field of mathematical modeling when solving various inverse problems in geophysics, astrophysics, scattering theory, and other research areas of natural sciences (see [1–6] for more details). A wide spectrum of applications of such equations motivates the growing interest in computational analysis of (1.1). Under the above conditions, equation (1.1) appears to be an ill-posed one [1–4], since the dependence of x^* on small variations of the operator F is not in general continuous. This means that small perturbations of F can result in considerable changes of x^* or even in the transformation of the original equation into an inconsistent problem. The circumstances outlined give rise to significant difficulties in practical solving of applied ill-posed problems of type (1.1) by traditional methods of computational mathematics. The characteristic feature of these methods is that they generally do not use available information on the level of noise in input data of the problem. The need of approximate solving of practical ill-posed problems has initiated the development of special regularization methods that allow us, using an approximate operator $\tilde{F}$ and the information on a level of errors δ, to obtain such an approximation to x^* which tends to x^* as $\delta \to 0$. Unlike methods of classical computational mathematics, the regularization procedures substantially use the information on a level of errors in input data.

In many cases, it is convenient to construct regularization methods for equations of form (1.1) by the following scheme (see [4, 5]). Let $\mathbf{F}$ be a class of operators $F : X_1 \to X_2$ that contains both the exact and the approximate (noisy) operators in (1.1). At the first stage, it is assumed that the original operator F is available without errors. A parametric family of mappings $\Re_\alpha : \mathbf{F} \to X_1$ is constructed such that $\lim\limits_{\alpha\to 0} \Re_\alpha(\mathbf{F}) = x^*$. The mapping $\Re_\alpha$ takes an operator $F \in \mathbf{F}$ and a regularization parameter $\alpha \in (0, \alpha_0]$ and, then, relates an approximate solution $x_\alpha = \Re_\alpha(F)$ for (1.1) to them. At the second stage, it is supposed that an approximate operator $\tilde{F} \in \mathbf{F}$ (rather than the exact operator F) and a value (vector) δ of estimates for the level of errors with respect to a suitable metric are available. Then, a dependence $\alpha = \alpha(\delta)$ (the rule of regularization parameter choice) is sought such that for the elements $x^\delta_{\alpha(\delta)} = \Re_{\alpha(\delta)}(\tilde{F})$ we have $\lim\limits_{\delta\to 0} x^\delta_{\alpha(\delta)} = x^*$. By the latter relation, the element $x^\delta_{\alpha(\delta)}$ may be taken as a desired approximation for the exact solution x^* adequate to a noisy operator $\tilde{F}$.

Another very actual and fruitful line of investigations within the regularization theory is related to studying classes of solutions x^* such that the following power estimates for the rate of convergence hold:

$$\|x_\alpha - x^*\|_{X_1} \leqslant c\alpha^p \quad (p > 0) \quad \forall \alpha \in (0, \alpha_0] \tag{1.2}$$

$$\|x^\delta_{\alpha(\delta)} - x^*\|_{X_1} \leqslant c\delta^p \quad (p > 0) \quad \forall \delta \in (0, \delta_0] \tag{1.3}$$

Throughout this paper, $\|\cdot\|_X$ denotes a norm of a Banach space X; $R(A) = \{y \in X_2 : y = Ax,\, x \in X_1\}$ and $N(A) = \{x \in X_1 : Ax = 0\}$ are the range and the nullspace of an operator $A \in L(X_1, X_2)$, respectively; A^* is the conjugate operator for A.

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The commonly accepted way to formalize requirements on the solution x^* that guarantee estimates (1.2) and (1.3) is to impose on x^* some sourcewise representation conditions of the form

$$x^* - \xi \in R((F'(x^*)^* F'(x^*))^p) \tag{1.4}$$

or

$$x^* - \xi \in R(F'(x^*)^p) \tag{1.5}$$

The element $\xi \in X_1$ used in (1.4) and (1.5) is a parameter of regularization methods, which can be used to control the convergence of the methods. Specifically, the choice $\xi = 0$ is, as a rule, admissible. This technique has received the most study for linear equations in Hilbert spaces, i.e., when the operator F in (1.1) is of the form

$$F(x) = Ax - f \tag{1.6}$$

where $A : X_1 \to X_2$ is a linear continuous operator, $f \in X_2$, and X_1 and X_2 are Hilbert spaces. Over the past thirty years, broad classes of regularization procedures for linear operator equations have been presented and studied, and procedures optimal on several classes of sourcewise representable solutions were distinguished [3–9]. In [4, 5, 10 – 12], the studies of extensions of these procedures for linear equations in Banach spaces have been initiated. An abstract scheme of constructing iterative regularization methods for nonlinear equations in Hilbert spaces was proposed in [5, 13 – 15]. We emphasize that the majority of papers within the above-mentioned framework are focussed on obtaining various estimates of convergence rate for given methods on the basis of representations (1.4) and (1.5), but the question of whether these conditions are necessary is not discussed as a rule. The presentation of a formalism allowing to derive necessary conditions for convergence of methods for linear and nonlinear ill-posed operator equations is among of the main aims of this paper. An application of this formalism to classes of the methods under consideration shows that conditions (1.4) and (1.5) used previously by most authors as sufficient ones for estimates (1.2) in fact appear to be very close to necessary conditions for (1.2). In other words, relations (1.4) and (1.5) yield a highly accurate description of the class of possible solutions for (1.1) such that power estimate (1.2) holds. In this regard, the results presented are similar to classical converse theorems of the Bernstein type in the theory of approximation [16]. Another purpose of this work is to continue studies of [4, 5] and to justify a general scheme of constructing regularization procedures for linear operator equations in Banach spaces. On this basis, we derive a technique to study necessary and sufficient conditions for the convergence of iterative methods of the Newton type for nonlinear equations in Banach spaces; this thechnique generalizes formalism previously developed in [5, 13 – 15] for the case of Hilbert spaces.

Our paper is organized as follows. The first part (Subsections 1.2 – 1.6) considers linear equations (1.1) with operators F of type (1.6). In Subsection 2 we rcall a well-known scheme [4, 8] of constructing regularization methods for linear ill-posed equations in Hilbert spaces. These constructions may be considered as a starting point for our further examinations. In Subsection 3 we give a survey of recent results on the sufficiency of sourcewise representation conditions for convergence of the above-mentioned methods with the power rate of convergence. Theorems 3.1 and 3.2 from this section have been obtained in cooperation with N. A. Yusoupova. In Subsection 4 we justify an extension of the scheme from Subsection 2 for linear equations in Banach spaces when input data are given without errors. In this case, it is convenient to take the sourcewise representation condition in form (1.5). We prove that, in Banach spaces, relation (1.5) is sufficient for (1.2) with the same exponent p. In Subsection 5 we establish the regularization property for the scheme from Subsection 4 when input data are noisy. Finally, in Subsection 6 we prove that sourcewise representation (1.5) in Banach spaces is actually very close to a sufficient condition for (1.2). The second part of this paper will be concerned with iterative methods of solving nonlinear equations (1.1).

2. Regularization methods for linear equations in Hilbert spaces. In this section we present a basic scheme of constructing regularization methods for linear operator equations in Hilbert spaces (see [4, 8]).

First, we recall the necessary notations and definitions. As usual, for Banach spaces X_1 and X_2 we denote by $L(X_1, X_2)$ the space of all linear continuous operators $A: X_1 \to X_2$ supplied with the norm $\|A\|_{L(X_1,X_2)} = \max\{\|Ax\|_{X_2}: \|x\|_{X_1} \leqslant 1\}$. For simplicity, we put $L(X, X) \equiv L(X)$. By $\sigma(A)$ and $\rho(A) = \mathbf{C}\backslash\sigma(A)$ we denote the spectrum and the resolvent set of an operator A, respectively; $A \in L(X)$, $R(\lambda, A) = (\lambda E - A)^{-1}$ is the resolvent operator for A and E is the identity operator.

Consider the equation

$$Ax = f, \quad x \in X_1 \tag{2.1}$$

where $A \in L(X_1, X_2), f \in X_2$, and X_1 and X_2 are Hilbert spaces.

Since the class of selfadjoint operators possesses a highly developed operator calculus (see, for example, [17]), equations involving such operators are, in many respects, more convenient to deal with. If the original operator A is not selfadjoint, then, acting on (2.1) by the conjugate operator A^*, we come to the equation

$$A^*Ax = A^*f, \quad x \in X_1 \tag{2.2}$$

with the selfadjoint operator A^*A. Denote by P the orthoprojector on the closure $\mathrm{cl}\, R(A)$ of the range $R(A)$. The relation between solutions of equations (2.1) and (2.2) establishes the following well-known assertion.

Lemma 2.1. *The solution set of* (2.2) *coincides with the solution set of the equation*

$$Ax = Pf, \quad x \in X_1$$

The solutions of (2.2) are called the quasisolutions for (2.1). Note that if $f \notin R(A)$ but $Pf \in R(A)$, then equation (2.1) has no solutions but possesses a nonempty set of quasisolutions. The set of quasisolutions being not empty obviously coincides with the solution set. In this section we consider the problem of finding a solution of (2.1) in the case of a selfadjoint operator $A^* = A \in L(X)$, $X = X_1 = X_2$, and a quasisolution when the operator A is not selfadjoint. Denote by X^* (X_1^*) the set of all solutions (quasisolutions) of equation (2.1). In what follows, we assume that $X^* \neq \emptyset$ ($X_1^* \neq \emptyset$).

We define the class $\mathbf{F}$ of possible exact and approximate data (A, f) in (2.1) as $\mathbf{F} = L(X_1, X_2) \times X_2$.

Let us turn now to the structure of mappings $\Re_\alpha\colon \mathbf{F} \to X_1$, $\alpha \in (0, \alpha_0]$, that generate regularization methods for equations (2.1). Recall that functions of a selfadjoint operator $A \in L(X)$ with its spectrum in $[M_0, M]$ can be defined with the use of the spectral decomposition $A = \int_{M_0}^{M} \lambda dE_\lambda$ [17, Ch. IX], where $\{E_\lambda\}$ is the family of spectral projectors for the operator A. In these notations, let the function $\varphi\colon [M_0, M] \to \mathbf{C}$ be measurable, finite, and defined almost everywhere with respect to the family $\{E_\lambda\}$ (i.e., with respect to all Lebesgue–Stieltjes measures generated by the functions $\|E_\lambda x\|_X^2$, $x \in X$). Then, the following representations hold

$$\varphi(A) = \int_{M_0}^{M} \varphi(\lambda)\, dE_\lambda \tag{2.3}$$

$$\|\varphi(A)x\|_X^2 = \int_{M_0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|_X^2, \quad x \in D(\varphi(A)) \tag{2.4}$$

In the case of a real-valued function $\varphi(\lambda)$, the operator $\varphi(A)$ is selfadjoint along with A; $\varphi(\lambda)$ is bounded if and only if $\operatorname{vrai}\sup_{\{E_\lambda\}} |\varphi(\lambda)| < \infty$ with an upper bound calculated over the family of all measures generated by the projectors $\{E_\lambda\}$. Moreover,

$$\|\varphi(A)\|_{L(X)} = \operatorname{vrai}\sup_{\{E_\lambda\}} |\varphi(\lambda)| \tag{2.5}$$

Generally, the operator $\varphi(A)$ is unbounded and its domain of definition

$$D(\varphi(A)) = \Big\{x \in X\colon \int_{M_0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|_X^2 < \infty\Big\} \tag{2.6}$$

is dense in X.

First, let us consider the case when $X_1 = X_2 = X$ and $A \in L(X)$ is a selfadjoint operator. Choose an initial guess $\xi \in X$ for x^* and fix a family $\Theta(\lambda, \alpha)$, $\alpha \in (0, \alpha_0]$, of Borel measurable real- or complex-valued functions on $[M_0, M] \supset \sigma(A)$ that satisfy the following condition.

Assumption 2.1. For all $p \in [0, p_0]$ ($p_0 > 0$)

$$\sup_{\lambda \in [M_0, M]} |\lambda|^p\, |1 - \Theta(\lambda, \alpha)\, \lambda| \leqslant c_0 \alpha^p \quad \forall \alpha \in (0, \alpha_0] \tag{2.7}$$

where $c_0 = c_0(p_0) > 0$.

Throughout the paper, we denote by $c_0, c_1, \ldots$ positive constants, which may depend on a given equation and characteristics of the methods under consideration. We put

$$\Re_\alpha(A, f) = (E - \Theta(A, \alpha)\, A)\, \xi + \Theta(A, \alpha)\, f$$

By doing so, we have defined the family of procedures

$$x_\alpha = (E - \Theta(A, \alpha)\, A)\, \xi + \Theta(A, \alpha)\, f, \quad \alpha \in (0, \alpha_0] \tag{2.8}$$

for approximation of the solution to original equation (2.1). The functions $\Theta(\lambda, \alpha)$ are said to be generating functions for the group of methods (2.8). The next statement establishes the convergence and formulates estimates of convergence rate for methods (2.8).

Theorem 2.1 [4, pp. 33–37; 8, p. 42]. *Suppose Assumption* 2.1 *is satisfied. Then*

$$\lim_{\alpha\to 0} \|x_\alpha - x^*\|_X = 0 \tag{2.9}$$

where x^* is the solution of (2.1) *nearest to ξ, i.e.,*

$$x^* \in X^*, \quad \|x^* - \xi\|_X = \min\{\|x - \xi\|_X : x \in X^*\} \tag{2.10}$$

If, in addition, the initial discrepancy can be represented as

$$x^* - \xi = A^p v, \quad v \in X \tag{2.11}$$

where $p, q \geqslant 0$, $p + q \in (0, p_0]$, then along with (2.9) *we have*

$$\|A^q(x_\alpha - x^*)\|_X \leqslant c_1 \|v\|_X \alpha^{p+q} \quad \forall \alpha \in (0, \alpha_0] \tag{2.12}$$

Similarly to (2.8), taking (2.2) in place of (2.1), we obtain a regularizing mapping $\Re_\alpha$ when the operator A is not selfadjoint. Let the generating functions $\Theta(\lambda, \alpha)$ satisfy condition (2.7). We denote

$$\Re_\alpha(A, f) = (E - \Theta(A^*A, \alpha)\, A^*A)\, \xi + \Theta(A^*A, \alpha)\, A^* f$$

and construct the family of procedures for finding quasisolutions of equation (2.1) as

$$x_\alpha = (E - \Theta(A^*A, \alpha)\, A^*A)\, \xi + \Theta(A^*A, \alpha)\, A^* f, \quad \alpha \in (0, \alpha_0] \tag{2.13}$$

Theorem 2.2 [4, pp. 33–37; 8, p. 45]. *Suppose Assumption* 2.1 *is satisfied. Then*

$$\lim_{\alpha\to 0} \|x_\alpha - x^*\|_{X_1} = 0$$

where x^* is the quasisolution of (2.1) *nearest to ξ, i.e.,*

$$x^* \in X_1^*, \quad \|x^* - \xi\|_{X_1} = \min\{\|x - \xi\|_{X_1} : x \in X_1^*\} \tag{2.14}$$

If, in addition, the initial discrepancy can be represented

$$x^* - \xi = (A^*A)^p w, \quad w \in X_1 \tag{2.15}$$

where $p, q \geqslant 0$, $p + q \in (0, p_0]$, then

$$\|(A^*A)^q(x_\alpha - x^*)\|_{X_1} \leqslant c_2 \|w\|_{X_1} \alpha^{p+q} \quad \forall \in (0, \alpha_0] \tag{2.16}$$

Remark 2.1. If $q = 0$, then the left-hand sides of (2.12) and (2.16) involve the usual pointwise discrepancy $x_\alpha - x^*$ of the solution x^*. In the case of $q = 1$, the expressions under the norm signs take the form $A(x_\alpha - x^*) = Ax_\alpha - f$ and $A^*A(x_\alpha - x^*) = A^*Ax_\alpha - A^*f$, respectively, and represent the discrepancies of the right-hand sides in equations (2.1) and (2.2).

Note that, in nontrivial situations, assumptions (2.11) and (2.15) with $p' > p$ are more rigid as compared with the exponent $p > 0$. Therefore, these conditions define more amd more restrictive classes of initial discrepancies $x^* - \xi$ as p increases. Specifically, when X_1, X_2 are functional spaces (for eqxample, L_2 or W_2^k) and the operator A is integral, conditions (2.11) and (2.15) mean that the discrepancy $x^* - \xi$ possesses an increasing smoothness as compared with the usual smoothness of elements from X_1. In the special case when A is the Green operator for an elliptic differential operator, these conditions are equivalent to the inclusion $x^* - \xi \in C^r$ with the coefficient r of smoothness proportional to p ([18, p. 454]).

We now present a number of generating functions $\Theta(\lambda, \alpha)$ that are of the widest applications and specify the abstract scheme (2.8) for these functions. Note that in the case of ill-posed equation (2.1) we are interested in $0 \in \sigma(A)$. Consequently, for bounds of the segment $[M_0, M] \supset \sigma(A)$ we have $M_0 \leqslant 0$, $M \geqslant 0$.

Example 2.1. Let $M_0 = 0$. Then the function

$$\Theta(\lambda, \alpha) = (\lambda + \alpha)^{-1} \tag{2.17}$$

satisfies (2.7) for all $p_0 \in (0, 1]$. A practical implementation of scheme (2.8) with generating function (2.17) is equivalent to solving the equation

$$(A + \alpha E)x_\alpha = \alpha\xi + f$$

with the continuously invertible operator $A+\alpha E$. Method (2.8), (2.17) is known as the method of M. M. Lavrent'ev.

Example 2.2. Choose an arbitrary $N \in \mathbf{N}$, $\mathbf{N} \overset{\text{def}}{=} \{1,2,\dots\}$ and assume that $M_0=0$. Then the function

$$\Theta(\lambda,\alpha)=\frac{1}{\lambda}\Big[1-\Big(\frac{\alpha}{\lambda+\alpha}\Big)^N\Big] \tag{2.18}$$

satisfies (2.7) for all $p_0 \in (0,N]$. We see that function (2.17) belongs to family (2.18) with $N=1$. The computation of the approximation x_α from (2.8) can practically be performed by the following finite iterative process [8, p. 19]:

$$x_\alpha = x_\alpha^{(N)} \tag{2.19}$$

where

$$x_\alpha^{(0)}=\xi, \quad (A+\alpha E)x_\alpha^{(k+1)}=\alpha x_\alpha^{(k)}+f, \quad k=0,1,\dots,N-1 \tag{2.20}$$

Method (2.19), (2.20) is known as the iterated method of M. M. Lavrent'ev.

An analog of the family of generating functions (2.18) for not selfadjoint operators A looks like as follows.

Example 2.3. The function

$$\Theta(\lambda,\alpha)=\frac{1}{\lambda}\Big[1-\Big(\frac{\alpha i}{\lambda+\alpha i}\Big)^N\Big] \tag{2.21}$$

satisfies (2.7) for all $p_0 \in (0,N]$ and $M_0 \leqslant 0$. Method (2.8), (2.21) can be implemented similarly to (2.19) and (2.20) [8, p. 22]: $x_\alpha = x_\alpha^{(N)}$ with

$$x_\alpha^{(0)}=\xi, \quad (A+\alpha i E)x_\alpha^{(k+1)}=\alpha i x_\alpha^{(k)}+f, \quad k=0,1,\dots,N-1$$

Simple calculations prove that the upper bounds on p_0 indicated in Examples 2.1 – 2.3 cannot in general be relaxed. This means that Theorems 2.1 and 2.2 do not guarantee the convergence of approximations x_α with estimates (2.12) and (2.16) involving an exponent $p+q>p_0$ even if the value of p in representations (2.11) and (2.15) is arbitrarily large. This effect is known as the saturation phenomenon [4, 8, 13]. Hence, the rate of convergence of saturating methods increases with the exponent p until an appropriate threshold value is reached and, then, remains constant when p increases. Here are some examples of methods (2.8) free of the saturation.

Example 2.4. Assume that $M_0=0$. Then the function

$$\Theta(\lambda,\alpha)=\begin{cases}\dfrac{1}{\lambda}\left(1-e^{-\lambda/\alpha}\right), & \lambda\neq 0\\[2mm] \dfrac{1}{\alpha}, & \lambda=0\end{cases} \tag{2.22}$$

satisfies (2.7) for all $p_0>0$. Procedure (2.8), (2.22) can be implemented as follows [8, p. 27]:

$$x_\alpha = u(\alpha^{-1}), \quad \alpha\in(0,\alpha_0] \tag{2.23}$$

where $u=u(t)$ is the solution for the Cauchy problem

$$\frac{du}{dt}+Au=f, \quad u(0)=\xi \tag{2.24}$$

Example 2.5. Let $M_0=0$ and the function $g\colon[0,M]\to\mathbf{R}$ be Borel measurable, bounded, and continuous at the point $\lambda=0$. Also suppose that

$$\sup_{\lambda\in[\varepsilon,M]}|1-\lambda g(\lambda)|\leqslant 1 \quad \forall\varepsilon\in(0,M)$$

Then the function

$$\Theta(\lambda,\alpha)=\begin{cases}\dfrac{1}{\lambda}\left[1-(1-\lambda g(\lambda))\right]^{1/\lambda}, & \lambda\neq 0\\[2mm] \dfrac{g(0)}{\alpha}, & \lambda=0\end{cases} \tag{2.25}$$

defined on the discrete set of regularization parameters $\alpha=1,\frac{1}{2},\dots,\frac{1}{n},\dots$ satisfies (2.7) for all $p_0>0$. The above-obtained procedure (2.8), (2.25) can be implemented as a finite iterative process [8, p. 37]: if $\alpha=\frac{1}{n}$, then

$$x_\alpha = x^{(n)} \tag{2.26}$$

where

$$x^{(0)} = \xi; \quad x^{(k+1)} = x^{(k)} - g(A)\left(Ax^{(k)} - f\right), \quad k = 0, 1, \ldots, n-1 \tag{2.27}$$

The simplest examples of the functions $g(\lambda)$ that meet the above-mentioned conditions are the following: $g(\lambda) \equiv \mu_0 \in (0, \frac{2}{M})$ and $g(\lambda) = (\lambda + \mu_0)^{-1}$, $\mu_0 > 0$.

Since the functions of Examples 2.4 and 2.5 satisfy Assumption 1.1 without any upper bounds on p_0, regularization procedures (2.23), (2.24) and (2.26), (2.27) are actually not saturating. Specifications of (2.13) for generating the functions from Examples 1 – 5 can be easily written with A^*A and A^*f in place of A and f, respectively.

We now dwell briefly on the case of noisy data in (2.1) with a general not selfadjoint operator $A \in L(X_1, X_2)$. Assume that in place of the original data (A, f) in (2.1) only approximations $(A_h, f_\eta) \in L(X_1, X_2) \times X_2$ are available such that

$$\|A_h - A\|_{L(X_1,X_2)} \leqslant h, \quad \|f_\eta - f\|_{X_2} \leqslant \eta \tag{2.28}$$

Following (2.13), we construct an approximation for a quasisolution x^* as

$$x_{\alpha(h,\eta)}^{(h,\eta)} = \left(E - \Theta(A_h^* A_h, \alpha(h,\eta))\, A_h^* A_\eta\right) \xi + \Theta(A_h^* A_h, \alpha(h,\eta))\, A_h^* f_\eta$$

Without loss of generality, let $\sigma(A_h^* A_h), \sigma(A^*A) \subset [0, M]$. The next assertion establishes a possible way of coordination of the regularization parameter α and the levels of errors $\delta = (h, \eta)$ that guarantee the convergence of approximations $x_{\alpha(h,\eta)}^{(h,\eta)}$ to x^* as $h, \eta \to 0$.

Theorem 2.3. [4, p. 39; 8, p. 97]. *Suppose Assumption* 1.1 *is valid and*

$$\sup_{\lambda \in [0,M]} |\Theta(\lambda, \alpha)| \leqslant c_3 \alpha^{-1} \quad \forall \alpha \in (0, \alpha_0]$$

$$\lim_{\alpha \to 0} \sup_{\lambda \in [0,M]} \lambda\, |1 - \Theta(\lambda, \alpha)\lambda| = 0$$

Let the regularization parameter be coordinated with the levels of errors such that

$$\lim_{h,\eta \to 0} \alpha(h,\eta) = 0, \quad \lim_{h,\eta \to 0} \frac{h + \eta}{\alpha(h,\eta)} = 0$$

Then, uniformly in the choice of (A_h, f_η) *within conditions* (2.28) *we have*

$$\lim_{h,\eta \to 0} \left\| x_{\alpha(h,\eta)}^{(h,\eta)} - x^* \right\|_{X_1} = 0$$

with the quasisolution x^* *defined by* (2.14).

Moreover, if the initial discrepancy possesses representation (2.15) *and the regularization parameter is chosen such that* $\alpha(h,\eta) = c_4 (h + \eta)^{\frac{1}{p+1}}$, *then*

$$\left\| x_{\alpha(h,\eta)}^{(h,\eta)} - x^* \right\|_{X_1} \leqslant c_5 (h + \eta)^{\frac{p}{p+1}}$$

The latter estimate is unimprovable on the class of equations (2.1) *with solutions possessing representation* (2.15).

We refer to [3 – 6, 8] for further examples of rules for the regularization parameter choice.

3. Necessary conditions of convergence of regularization methods for linear equations in Hilbert spaces. In this section we prove under some appropriate assumptions on the generating functions $\Theta(\lambda, \alpha)$ that sourcewise representations (2.11) and (2.15) are very close to necessary conditions for methods (2.8) and (2.13) to converge with estimates (2.12) and (2.16). In addition to Assumption 2.1, we impose the following condition on $\Theta(\lambda, \alpha)$.

Assumption 3.1. Borel measurable functions $\Theta(\lambda, \alpha)$ satisfy Assumption 2.1, and there exists a constant $c_6 = c_6(\tau) > 0$ such that

$$\int_0^{\alpha_0} \alpha^{-2\tau - 1} |1 - \Theta(\lambda, \alpha)\lambda|^2 \, d\alpha \geqslant \frac{c_6}{|\lambda|^{2\tau}} \quad \forall \lambda \in [M_0, M] \backslash \{0\} \quad \forall \tau \in (0, p_0) \tag{3.1}$$

Theorem 3.1. *Suppose Assumption* 3.1 *is satisfied. For a fixed triplet* A, f, ξ *and given* p, q *such that* $p > 0$, $q \geqslant 0$, $p + q \in (0, p_0]$, *we assume that*

$$\|A^q (x_\alpha - x^*)\|_X \leqslant c_7 \alpha^{p+q} \quad \forall \alpha \in (0, \alpha_0] \tag{3.2}$$

with x_α and x^* defined by (2.8) *and* (2.10), *respectively. Then, for each $\varepsilon \in (0,p)$ the following inclusion holds:*

$$x^* - \xi \in R(A^{p-\varepsilon}) \tag{3.3}$$

Proof. From (2.8) and (3.2) with the use of the equality $f = Ax^*$, we obtain

$$\|A^q(x_\alpha - x^*)\|_X^2 = \|A^q(E - \Theta(A,\alpha)\,A)\,(x^* - \xi)\|_X^2 \leqslant c_7\alpha^{2(p+q)} \tag{3.4}$$

By (2.4) and (3.4), for each $\omega \in (0, 2p)$ we have

$$\int\limits_{[M_0,M]\backslash\{0\}} \alpha^{-2(p+q)-1+\omega}|\lambda|^{2q}|1 - \Theta(\lambda,\alpha)\lambda|^2\, d\|E_\lambda(x^* - \xi)\|_X^2 \leqslant c_7\alpha^{-1+\omega} \quad \forall\alpha \in (0, \alpha_0]$$

Integrating the both sides with respect to $\alpha \in (0,\alpha_0]$, we get

$$\int\limits_0^{\alpha_0}\int\limits_{[M_0,M]\backslash\{0\}} \alpha^{-2(p+q)-1+\omega}|\lambda|^{2q}|1 - \Theta(\lambda,\alpha)\lambda|^2\, d\|E_\lambda(x^* - \xi)\|_X^2\, d\alpha < \infty$$

Since the integrand is nonnegative, it follows from the Fubini theorem [19, p. 318] that

$$\int\limits_{[M_0,M]\backslash\{0\}} |\lambda|^{2q}\Big(\int\limits_0^{\alpha_0} \alpha^{-2(p+q)-1+\omega}|1 - \Theta(\lambda,\alpha)\lambda|^2 d\alpha\Big)\, d\|E_\lambda(x^* - \xi)\|_X^2 < \infty \tag{3.5}$$

Taking $\tau = p + q - \frac{\omega}{2}$ in (3.1), for the internal integral in (3.5) we obtain the estimate

$$\int\limits_0^{\alpha_0} \alpha^{-2(p+q)-1+\omega}|1 - \Theta(\lambda,\alpha)\lambda|^2\, d\alpha \geqslant \frac{c_6}{|\lambda|^{2(p+q)-\omega}} \quad \forall\lambda \in [M_0, M]\backslash\{0\}$$

Now by (2.10), $x^* - \xi \perp N(A)$ and, hence, the function $\|E_\lambda(x^* - \xi)\|_X^2$ is continuous at $\lambda = 0$. Therefore, the corresponding Lebesgue–Stieltjes measure of the singleton $\{0\}$ equals to zero and by (3.5) we have

$$\int\limits_{M_0}^{M} |\lambda|^{-2(p-\frac{\omega}{2})}\, d\|E_\lambda(x^* - \xi)\|_X^2 = \int\limits_{[M_0,M]\backslash\{0\}} |\lambda|^{-2(p-\frac{\omega}{2})}\, d\|E_\lambda(x^* - \xi)\|_X^2 < \infty$$

By (2.6), $x^* - \xi \in R(A^{p-\frac{\omega}{2}})$. Since $\omega \in (0, 2p)$ may be chosen arbitrarily small, for each $\varepsilon \in (0,p)$ inclusion (3.3) holds. This completes the proof.

Example 3.1. Direct calculations prove that estimate (3.1) is valid for the generating functions from Examples 2.1, 2.2 and 2.4 if $M_0 = 0$. The functions from Example 2.3 satisfy (3.1) for all $M_0 \leqslant 0$.

We now turn to iterative methods (2.26), (2.27). Suppose the following assumption is satisfied.

Assumption 3.2. The function $g(\lambda)$ satisfies the conditions of Example 2.5 and there exists a constant $c_8 = c_8(\tau) > 0$ such that

$$\sum_{n=1}^{\infty} n^{2\tau-1}|1 - \lambda g(\lambda)|^{2n} \geqslant \frac{c_8}{\lambda^{2\tau}} \quad \forall\lambda \in (0, M] \quad \forall\tau > 0 \tag{3.6}$$

Theorem 3.2. *Suppose Assumption* 3.2 *is satisfied. For a fixed triplet A, f, ξ and given p, q such that $p > 0$, $q \geqslant 0$, $p + q \in (0, p_0]$, we assume that*

$$\|A^q(x^{(n)} - x^*)\|_X \leqslant c_9 n^{-(p+q)} \quad \forall n \in \mathbf{N} \tag{3.7}$$

with $x^{(n)} = x_\alpha$ and x^* defined by (2.26), (2.27), *and* (2.10), *respectively. Then, for each $\varepsilon \in (0,p)$, inclusion* (3.3) *holds.*

Proof. Using (3.7), we get for all $n \in \mathbf{N}$ that

$$\|A^q(x^{(n)} - x^*)\|_X^2 = \|A^q(E - Ag(A))^n\,(x^* - \xi)\|_X^2 = \int\limits_0^M \lambda^{2q}(1 - \lambda g(\lambda))^{2n}\, d\|E_\lambda(x^* - \xi)\|_X^2 \leqslant \frac{c_9}{n^{2(p+q)}} \tag{3.8}$$

The next arguing follows from the proof of Theorem 3.1 with the summation over all $n \in \mathbf{N}$ in place of the integration with respect to $\alpha \in (0, \alpha_0]$. From (3.8) it follows that

$$\int_0^M n^{2(p+q)-1-\omega} \lambda^{2q} |1 - \lambda g(\lambda)|^{2n} \, d\|E_\lambda(x^* - \xi)\|_X^2 \leqslant \frac{c_9}{n^{1+\omega}}$$

for each $\omega \in (0, 2p)$. Summing up the above-obtained inequalities, we get

$$\sum_{n=1}^{\infty} \int_0^M n^{2(p+q)-1-\omega} \lambda^{2q} |1 - \lambda g(\lambda)|^{2n} \, d\|E_\lambda(x^* - \xi)\|_X^2 < \infty$$

The application of Levi's theorem [19, p. 305] with $\tau = p + q - \frac{\omega}{2}$ in (3.6) yields

$$\sum_{n=1}^{\infty} \int_0^M n^{2(p+q)-1-\omega} \lambda^{2q} |1 - \lambda g(\lambda)|^{2n} d\|E_\lambda(x^* - \xi)\|_X^2$$

$$= \int_0^M \lambda^{2q} \Big(\sum_{n=1}^{\infty} n^{2(p+q)-1-\omega} |1 - \lambda g(\lambda)|^{2n} \Big) d\|E_\lambda(x^* - \xi)\|_X^2 \geqslant c_8 \int_0^M \lambda^{-2(p-\omega/2)} \, d\|E_\lambda(x^* - \xi)\|_X^2$$

Now, by (2.6), $x^* - \xi \in R(A^{p-\varepsilon})$ $\forall \varepsilon \in (0, p)$. This completes the proof.

Remark 3.1. Theorem 3.2 remains true if inequality (3.6) is satisfied only for $\lambda \in (0, M_1]$ with $M_1 \in (0, M]$. This note is useful for practical verification of Assumption 3.2.

Example 3.2. It can easily be checked that Assumption 3.2 is satisfied for the functions $g(\lambda) \equiv \mu_0$, $\mu_0 \in (0, \frac{2}{M})$, and $g(\lambda) = (\lambda + \mu_0)^{-1}$, $\mu_0 > 0$, introduced in Example 2.5.

Remark 3.2. Examples from [6, 19] prove that Theorems 3.1 and 3.2 with the equality $\varepsilon = 0$ in place of the inclusion $\varepsilon \in (0, p)$ are not true in general. At the same time, they remain valid with $\varepsilon = 0$ in the case when $p = p_0^*$, where p_0^* is a maximum of all p_0 such that Assumption 2.1 holds (see [6]). The value of p_0^* is called the qualification of method (2.8) or (2.13). According to Section 1, functions (2.18) and (2.21) generate methods with the qualification $p_0^* = N$, whereas functions (2.22) and (2.25) yield methods with $p_0^* = \infty$.

When the operator $A \in L(X_1, X_2)$ is not selfadjoint, similar results can easily be formulated and proved with A^*A and A^*f in place of A and f.

To conclude this section, we note that the unimprovability of estimates for rates of convergence of methods (2.8) and (2.13) subject to conditions (2.11) and (2.15) uniformly in input data (A, f) and an initial guess ξ have been established by many authors [3, 4, 8, 21]. Contrary to these results, Theorems 3.1 and 3.2 deal with individual equations with a fixed initial guess ξ rather than with classes of problems. They show that the guaranteed order of rate of convergence completely depends on a priori information on the exponent in the sourcewise representation of a solution or quasisolution. The above formalism of deducing necessary conditions for qualified convergence of procedures (2.8) and (2.13) develops the technique of [20, Ch. I, § 9]. For the case when $q = 0$, the conclusions of these theorems have been obtained earlier in [6] by other means.

4. A class of methods for solving linear equations in Banach spaces. Suppose X is a complex Banach space. Given $A \in L(X)$ and $f \in X$, consider a linear equation

$$Ax = f, \quad x \in X \tag{4.1}$$

Assume that the solution set X^* of (4.1) is nonempty. An immediate extension of the Hilbert space technique presented above for Banach spaces turns out to be difficult mainly because of the lack of suitable generalization of the spectral decomposition formalism and of the corresponding operator calculus for sufficiently broad classes of operators $A \in L(X)$. The use of Riesz–Dunford operator calculus seems to be the most convenient way of extension of the previous results for Banach spaces. Suppose Γ is a positively oriented contour on the complex plane $\mathbf{C}$ such that Γ surrounds the spectrum $\sigma(A)$. Let the function $\varphi(\lambda)$ be analytic on an open neighborhood $D \supset \sigma(A)$, where $\Gamma \subset D$. Then, the function $\varphi(A)$ of an operator A can be defined by the Riesz–Dunford formula [17, p. 455]

$$\varphi(A) = \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda) R(\lambda, A) \, d\lambda \tag{4.2}$$

where the integral exists in Bochner's sense.

In [10, 11], the following extension of the process (2.8) for equation (4.1) in the Banach space X was proposed:

$$x_\alpha = (E - \Theta(A,\alpha)\,A)\,\xi + \Theta(A,\alpha)\,f, \quad \alpha \in (0,\alpha_0] \tag{4.3}$$

with $\Theta(A,\alpha)$ defined by (4.2). The meaning of the element $\xi \in X$ in (4.3) is the same as in (2.8). Under appropriate assumptions on the operator A and under condition (2.11) with $p = 1$, it was established in [10, 11] that approximations x_α converge to the solution x^* from (2.11) as $\alpha \to 0$; the case of a noisy right-hand side f in (4.1) was also investigated (see [4, pp. 50–54; 12]). We emphasize that the assumption of analiticity on the generating functions $\Theta(\lambda,\alpha)$ do not lead in practice to the loss of generality as compared with the Hilbert case, since the most useful schemes (4.3) are generated by analytic functions $\Theta(\lambda,\alpha)$ (see Examples 2.1–2.5). The aim of Sections 4–6 is to continue the study of regularization scheme (4.3) and to obtain analogs of the above results concerning methods (2.8) for processes (4.3). We shall first prove estimates for the rate of convergence of approximations x_α to x^* as $\alpha \to 0$ for arbitrary exponents $p > 0$ in the sourcewise representation. Then, regularization properties of methods (4.3) for the case of both the operator and right-hand side available with errors will be established. Finally, we shall prove analogs of Theorems 3.1 and 3.2 for (4.3). Note that in the most interesting case of ill-posed equation (4.1) we have $0 \in \sigma(A)$. Therefore, the power A^p cannot be defined directly by (4.2) except for exponents $p \in \mathbf{N}$, $\mathbf{N} \overset{\text{def}}{=} \{1,2,\ldots\}$, since the function λ^p is not analytic in a neighborhood of $\lambda = 0$. For the definition and properties of fractional powers A^p, $p > 0$, of operators $A \in L(X)$ with $0 \in \sigma(A)$ we refer to [22; 23, p. 156].

As in [4, p. 51; 10, 11], we restrict ourselves to the operators A that satisfy the following condition.

Assumption 4.1. There is $\varphi_0 \in (0,\pi)$ such that

$$\sigma(A) \subset K(\varphi_0), \quad K(\varphi_0) \overset{\text{def}}{=} \{\lambda \in \mathbf{C}\colon |\arg\lambda| \leqslant \varphi_0\} \tag{4.4}$$

and

$$\|R(\lambda,A)\|_{L(X)} \leqslant \frac{c_{10}}{|\lambda|} \quad \forall \lambda \in \mathbf{C}\backslash K(\varphi_0) \tag{4.5}$$

Fix a constant $R_0 > \|A\|_{L(X)}$. It is easy to show that $\sigma(A) \subset K(R_0,\varphi_0)$ with

$$K(R_0,\varphi_0) \overset{\text{def}}{=} K(\varphi_0) \cap S(R_0), \quad S(r) \overset{\text{def}}{=} \{\lambda \in \mathbf{C}\colon |\lambda| \leqslant r\}, \quad r > 0$$

Moreover, we obviously have an estimate similar to (4.5) with the sector $K(R_0,\varphi_0)$ in place of the cone $K(\varphi_0)$.

Let us recall necessary definitions from the theory of fractional powers of linear operators. Given a natural p, the power A^p is defined in the usual way, i.e., $A^p = A\cdots A$ (p times). For an operator A satisfying Assumption 4.1 and an arbitrary noninteger $p > 0$, we define A^p as follows.

Definition 4.1 [22; 23, p. 156]. For each $\mu \in (0,1)$

$$A^\mu \overset{\text{def}}{=} \frac{\sin \pi\mu}{\pi} \int\limits_0^\infty t^{\mu-1}(tE + A)^{-1}A\,dt \tag{4.6}$$

Let $p > 0$ and $m \in \mathbf{N}$ be such that $p \in (m, m+1)$. Then

$$A^p \overset{\text{def}}{=} A^m \cdot A^{p-m} \equiv A^{p-m} \cdot A^m$$

Note that by (4.5) the integral in (4.6) exists in Bochner's sense and represents an operator $A^\mu \in L(X)$. Given an operator $A \in L(X)$, consider its regularization $A_\varepsilon = A + \varepsilon E$. Let Assumption 4.1 be satisfied. Then, for all $\varepsilon > 0$ and $p > 0$ the power A^p_ε can be defined by formula (4.2) with $\varphi(\lambda) = \lambda^p$ provided that the contour γ surrounds the spectrum $\sigma(A_\varepsilon) = \{\lambda + \varepsilon\colon \lambda \in \sigma(A)\}$ and does not include the point $\lambda = 0$.

Lemma 4.1 [23, p. 155]. *Let an operator $A \in L(X)$ satisfy Assumption* 4.1. *Then, for each $\mu \in (0,1)$*

$$\|A^p_\varepsilon - A^p\|_{L(X)} \leqslant c_2\varepsilon^p \quad \forall \varepsilon > 0 \tag{4.7}$$

with a constant c_{11} that depends on A and μ only.

Let us now specify the class of generating functions $\Theta(\lambda,\alpha)$ that will be used in subsequent examinations. Suppose the following assumption is satisfied.

Assumption 4.2. For each $\alpha \in (0,\alpha_0]$ the function $\Theta(\lambda,\alpha)$ is analytic in λ on an open subset $D_\alpha \subset \mathbf{C}$ such that

$$K_\alpha(R_0,d_0,\varphi_0) \subset D_\alpha \tag{4.8}$$

with

$$K_\alpha(R_0, d_0, \varphi_0) \stackrel{\text{def}}{=} K(R_0, \varphi_0) \cup S_{\min\{R_0, d_0\alpha\}}(0)$$

and a fixed constant $d_0 \in (0, 1)$.

If needed, in the course of our examinations we shall impose additional conditions on $\Theta(\lambda, \alpha)$. By $\operatorname{int} G$ and $\operatorname{fr} G = \operatorname{cl} G \backslash \operatorname{int} G$ we denote the interior and the boundary of a subset $G \subset \mathbf{C}$, respectively; $\operatorname{cl} G$ is the closure of G. We put

$$\gamma_\alpha = \operatorname{fr} K_\alpha(R_0, d_0, \varphi_0), \quad \alpha > 0$$

It follows from (4.4) and (4.8) that the operator $\Theta(A, \alpha)$ admits representation (4.2) with $\varphi(\lambda) = \Theta(\lambda, \alpha)$ if $\Gamma = \Gamma_\alpha$ is taken in such a way that $\Gamma_\alpha \subset D$ and γ_α lies inside the contour Γ_α with both the contours oriented positively. In other words, we have the representation

$$\Theta(A, \alpha) = \frac{1}{2\pi i} \int\limits_{\Gamma_\alpha} \Theta(\lambda, \alpha) R(\lambda, A)\, d\lambda, \quad \alpha \in (0, \alpha_0] \tag{4.9}$$

with Γ_α chosen as outlined above. In addition, assume that the contour Γ_α does not surround the point $\lambda = -d_1\alpha$ $(d_1 > d_0)$ $\forall \alpha \in (0, \alpha_0]$. The application of Assumption 4.2 yields that such families of contours Γ_α, $\alpha \in (0, \alpha_0]$ do exist.

Suppose the initial discrepancy possesses the sourcewise representation

$$x^* - \xi = A^p v, \quad v \in X \quad x^* \in X^*, \quad p > 0 \tag{4.10}$$

Then, from (4.3) for each $q \geqslant 0$ we have

$$A^q(x_\alpha - x^*) = -(E - \Theta(A, \alpha)\, A)\, A^{p+q} v \tag{4.11}$$

Denote by $m = [p+q]$ and $\mu = p+q-m$ the integer and fractional parts of $p+q$, respectively. By the definition given above, $A^{p+q} = A^m \cdot A^\mu$. By (4.11), for each $\varepsilon > 0$ we get

$$\|A^q(x_\alpha - x^*)\|_X \leqslant \|(E - \Theta(A, \alpha)\, A)\, A^m\, (A + \varepsilon E)^\mu v\|_X + \|(E - \Theta(A, \alpha)\, A) A^m\, [(A + \varepsilon E)^\mu - A^\mu] v\|_X \tag{4.12}$$

Let us put $\varepsilon = c_{12}\alpha$ and estimate the summands in the right-hand side of inequality (4.12). From (4.2) we obtain

$$\begin{aligned} \|(E - \Theta(A, \alpha)A) A^m (A + \varepsilon E)^\mu v\|_X &\leqslant \frac{1}{2\pi} \|v\|_X \cdot \int\limits_{\Gamma_\alpha} |1 - \Theta(\lambda, \alpha)\lambda| |\lambda|^m |\lambda + \varepsilon|^\mu \|R(\lambda, A)\|_{L(X)}\, |d\lambda| \\ &\leqslant c_{13} \|v\|_X \cdot \int\limits_{\Gamma_\alpha} |1 - \Theta(\lambda, \alpha)\lambda|\, (|\lambda|^{p+q-1} + \alpha^\mu |\lambda|^{m-1})\, |d\lambda| \end{aligned} \tag{4.13}$$

Having this in mind, we now impose one more restriction on the generating functions $\Theta(\lambda, \alpha)$ (compare with Assumption 2.1).

Assumption 4.3. There exists a constant c_{14} independent of α such that for all $p \in [0, p_0]$ $(p_0 > 0)$

$$\int\limits_{\Gamma_\alpha} |1 - \Theta(\lambda, \alpha)\lambda| |\lambda|^{p-1} |\, d\lambda| \leqslant c_{14} \alpha^p \quad \forall \alpha \in (0, \alpha_0] \tag{4.14}$$

Suppose Assumption 4.3 is valid and $p+q \in (0, p_0]$. Then, by (4.13) and (4.14) with the use of the estimate $|\lambda| \geqslant d_0\alpha$ $\forall \lambda \in \Gamma_\alpha$ we obtain

$$\|(E - \Theta(A, \alpha)\, A)\, A^m (A + \varepsilon E)^\mu v\|_X \leqslant c_{15} \|v\|_X \alpha^p \quad \forall \alpha \in (0, \alpha_0] \tag{4.15}$$

By (4.5) and (4.7), for the second term in the right-hand side of (4.12) we get

$$\begin{aligned} \|(E - \Theta(A, \alpha)A) A^m [(A + \varepsilon E)^\mu - A^\mu] v\|_X &\leqslant c_{11} \|v\|_X \varepsilon^\mu \|(E - \Theta(A, \alpha)A) A^m\|_{L(X)} \\ &\leqslant \frac{c_{11}}{2\pi} \|v\|_X \varepsilon^\mu \int\limits_{\Gamma_\alpha} |1 - \Theta(\lambda, \alpha)\lambda| |\lambda|^m \|R(\lambda, A)\|_{L(X)}\, |d\lambda| \leqslant c_{16} \|v\|_X \alpha^p \quad \forall \alpha \in (0, \alpha_0] \end{aligned} \tag{4.16}$$

Estimates (4.12), (4.15), and (4.16) yield

$$\|A^q(x_\alpha - x^*)\|_X \leqslant c_{17} \|v\|_X \alpha^p \quad \forall \alpha \in (0, \alpha_0]. \tag{4.17}$$

Hence, we have proved the following statement.

Theorem 4.1. *Suppose Assumptions* 4.1 — –4.3 *are satisfied and sourcewise representation* (4.10) *with* $q \geqslant 0$, $p+q \in (0, p_0]$, *holds. Then, estimate* (4.17) *is valid with the solution* x^* *from* (4.10).

Remark 4.1. It follows from Theorem 4.1 that the solution x^* to (4.1) which satisfies (4.10) is unique.

Remark 4.2. The conclusion of Theorem 4.1 remains valid in a slightly more general situation when the cone $K(\varphi_0)$ in Assumption 4.1 is replaced by $\tilde{K}(\varphi_0) = \{\lambda \in \mathbf{C}\colon |\arg\lambda - \lambda_0| \leqslant \varphi_0\}$, $\lambda_0 \in (0, 2\pi)$. It is easy to see that the multiplication of both sides of equation (4.1) by $e^{-i\lambda_0}$ reduces this case to the above-studied one.

Concluding this section, we note that Assumption 4.1 is satisfied with an appropriate $\varphi_0 \in (0, \pi)$ for the classes of operators A listed below.

i) Selfadjoint nonnegative operators $A^* = A \geqslant 0$ in a Hilbert space X. Assumption 4.1 holds with $c_{10} = 1$, $\varphi_0 \in (0, \pi)$.

ii) Accretive operators in a Hilbert space, i.e., the operators that satisfy [24, p. 350]

$$\mathrm{Re}\,(Ax, x) \geqslant 0 \quad \forall x \in X$$

Assumption 4.1 is valid for each $\varphi_0 \in (\frac{\pi}{2}, \pi)$.

iii) Spectral operators of the scalar type in a Banach space with a spectrum from the cone $K(\psi_0)$, $\psi_0 \in (0, \pi)$ (see [25, p. 41]). A value of $\varphi_0 \in (\psi_0, \pi)$ is arbitrary.

iv) Operators A in a Banach space X such that $\sigma(A) \subset K(\psi_0)$, $\psi_0 \in (0, \pi)$, and

$$\|R(-t, A)\|_{L(X)} \leqslant \frac{c_{18}}{t}, \quad c_{18} > 1 \quad \forall t > 0$$

In this case, Assumption 4.1 is valid for all $\varphi_0 \in (\psi_1, \pi)$ with

$$\psi_1 = \max\Big\{\psi_0, \pi - \arcsin\frac{1}{c_{18}}\Big\}$$

5. A class of regularization algorithms for linear equations in Banach spaces. Suppose now that the input data in (4.1) are available with errors, i.e., instead of the original operators A and f their approximations $(A_h, f_\eta) \in \mathbf{F}$, $\mathbf{F} = L(X) \times X$ are given such that

$$\|A_h - A\|_{L(X)} \leqslant h, \quad \|f_\eta - f\|_X \leqslant \eta \tag{5.1}$$

An upper estimate $\delta = (h, \eta)$ for errors in input data is also assumed to be given. Following (4.3), we shall construct an approximation for a solution of (4.1) as

$$x_{\alpha(h,\eta)}^{(h,\eta)} = (E - \Theta(A_h, \alpha(h,\eta))\, A_h)\, \xi + \Theta(A_h, \alpha(h,\eta))\, f_\eta \tag{5.2}$$

The regularization parameter $\alpha = \alpha(h, \eta)$ in (5.2) should appropriately be coordinated with the levels of errors h, η in order to guarantee the regularization property

$$\lim_{h,\eta \to 0} \left\| x_{\alpha(h,\eta)}^{(h,\eta)} - x^* \right\|_X = 0 \quad (x^* \in X^*) \tag{5.3}$$

uniformly in $(A_h, f_\eta) \in \mathbf{F}$ subject to (5.1).

To justify (5.2), we first prove that the function $\Theta(A_h, \alpha)$ of the approximate operator A_h can be defined by (4.9). For this purpose, we recall the following well-known proposition.

Lemma 5.1 [23, p. 185; 26, p. 141]. *Let* $\lambda \in \rho(A)$, $A \in L(X)$, *and* $B \in L(X)$.

1) *Suppose* $\|BR(\lambda, A)\|_{L(X)} < 1$. *Then,* $\lambda \in \rho(A+B)$ *and the following representation holds:*

$$R(\lambda, A+B) = R(\lambda, A) \sum_{k=0}^{\infty} (BR(\lambda, A))^k \tag{5.4}$$

2) *Suppose* $\|R(\lambda, A)B\|_{L(X)} < 1$. *Then,* $\lambda \in \rho(A+B)$ *and*

$$R(\lambda, A+B) = \sum_{k=0}^{\infty} (R(\lambda, A)B)^k R(\lambda, A) \tag{5.5}$$

The series in (5.4) *and* (5.5) *converge absolutely with respect to the norm of* $L(X)$.

The next lemma establishes conditions sufficient for the contour Γ_α (see (4.9)) to surround the spectrum $\sigma(A_h)$ along with the spectrum $\sigma(A)$.

Lemma 5.2. *Suppose*

$$\frac{c_{10}h}{d_0\alpha} \leqslant \omega_0 \tag{5.6}$$

with a constant $\omega_0 \in (0,1)$. *Then, the contour* Γ_α *surrounds the spectrum* $\sigma(A_h)$. *Hence, for the operator* $\Theta(A_h,\alpha)$ *representation* (4.9) *holds.*

Proof. By construction of $K_\alpha(R_0,d_0,\varphi_0)$, for each $\lambda \in \mathbf{C}\backslash \operatorname{int} K_\alpha(R_0,d_0,\varphi_0)$ we have $|\lambda| \geqslant d_0\alpha$. Taking $B = A_h - A$ in Lemma 5.1, with the use of (5.1) and (5.6) we obtain

$$\|BR(\lambda,A)\|_{L(X)} \leqslant \frac{c_{10}h}{|\lambda|} \leqslant \frac{c_{10}h}{d_0\alpha} \leqslant \omega_0 < 1$$

Therefore, $\lambda \in \rho(A_h)$ and the contour Γ_α contains all the points of the spectrum $\sigma(A_h)$. This completes the proof.

Throughout this section, we suppose that condition (5.6) is satisfied.

Let a solution $x^* \in X^*$ possesses sourcewise representation (4.10). Note that

$$x_\alpha^{(h,\eta)} - x^* = (E - \Theta(A_h,\alpha)A_h)\,(\xi - x^*) + \Theta(A_h,\alpha)\,[(f_\eta - f) + (A - A_h)x^*] \tag{5.7}$$

From (4.10), (5.1), and (5.7) it follows that

$$\|x_\alpha^{(h,\eta)} - x^*\|_X \leqslant \|\Theta(A_h,\alpha)\|_{L(X)}\,(\eta + \|x^*\|_X h) + \|(E - \Theta(A_h,\alpha)A_h)\,A^p v\|_X \tag{5.8}$$

Let us estimate each summand in the right-hand side of (5.8).

By (5.6), (5.1), and (5.4), we have

$$\|R(\lambda,A_h)\|_{L(X)} \leqslant \|R(\lambda,A)\|_{L(X)} \cdot \sum_{k=0}^{\infty} \|(A_h - A)R(\lambda,A)\|_{L(X)}^k \leqslant \frac{c_{10}}{(1-\omega_0)|\lambda|} \quad \forall \lambda \in \Gamma_\alpha, \quad \alpha \in (0,\alpha_0]$$

From (4.9) we obtain

$$\|\Theta(A_h,\alpha)\|_{L(X)} \leqslant \frac{1}{2\pi}\int_{\Gamma_\alpha} |\Theta(\lambda,\alpha)|\,\|R(\lambda,A_h)\|_{L(X)}\,|d\lambda| \leqslant c_{19}\int_{\Gamma_\alpha} \frac{|\Theta(\lambda,\alpha)|}{|\lambda|}|\,d\lambda| \quad \forall \alpha \in (0,\alpha_0] \tag{5.9}$$

In addition to Assumptions 4.2 and 4.3, we now impose the following condition on the generating functions $\Theta(\lambda,\alpha)$.

Assumption 5.1. For each $\alpha \in (0,\alpha_0]$,

$$\int_{\Gamma_\alpha} \frac{|\Theta(\lambda,\alpha)|}{|\lambda|}\,|d\lambda| \leqslant \frac{c_{20}}{\alpha}$$

By (5.9) and Assumption 5.1, we get

$$\|\Theta(A_h,\alpha)\|_{L(X)} \leqslant \frac{c_{21}}{\alpha} \quad \forall \alpha \in (0,\alpha_0] \tag{5.10}$$

For the second term in the right-hand side of (5.8), we have the estimate

$$\|(E - \Theta(A_h,\alpha)\,A_h)\,A^p v\|_X \leqslant \|(E - \Theta(A,\alpha)\,A)\,A^p v\|_X + \|(\Theta(A_h,\alpha)\,A_h - \Theta(A,\alpha)\,A)\,A^p v\|_X \tag{5.11}$$

As in the proof of Theorem 4.1, from (4.14) we come to the inequality

$$\|(E - \Theta(A,\alpha)\,A)\,A^p v\|_X \leqslant c_{22}\|v\|_X\alpha^p \quad \forall \alpha \in (0,\alpha_0] \tag{5.12}$$

Denote $m = [p]$, $\mu = p - m$, $\varepsilon = c_{23}\alpha$. Then

$$\begin{aligned}\|(\Theta(A_h,\alpha)A_h - \Theta(A,\alpha)A)A^p v\|_X &\leqslant \|(\Theta(A_h,\alpha)A_h - \Theta(A,\alpha)A)A^m(A+\varepsilon E)^\mu v\|_X \\ &+ \|(\Theta(A_h,\alpha)A_h - \Theta(A,\alpha)A)A^m[(A+\varepsilon E)^\mu - A^\mu]v\|_X\end{aligned} \tag{5.13}$$

For the first term in (5.13) we have

$$\begin{aligned}&\|(\Theta(A_h,\alpha)A_h - \Theta(A,\alpha)A)A^m(A+\varepsilon E)^\mu v\|_X \\ &\quad\leqslant \frac{1}{2\pi}\|v\| \cdot \int_{\Gamma_\alpha} |1 - \Theta(\lambda,\alpha)\lambda| \cdot \|(R(\lambda,A) - R(\lambda,A_h))A^m(A+\varepsilon E)^\mu\|_{L(X)}\,|d\lambda|\end{aligned}$$

The application of Lemma 5.1 (see (5.5)) and (5.6) yields

$$\|(R(\lambda, A) - R(\lambda, A_h))A^m(A+\varepsilon E)^\mu\|_{L(X)} \leqslant \sum_{k=1}^{\infty}(\|R(\lambda, A)\|_{L(X)}h)^k\|R(\lambda, A)A^m\|_{L(X)}\|(A+\varepsilon E)^\mu\|_{L(X)}$$
$$\leqslant \frac{c_{24}h}{(1-\omega_0)|\lambda|}\|R(\lambda, A)A^m\|_{L(X)} \quad \forall\lambda\in\Gamma_\alpha, \quad \alpha\in(0,\alpha_0]$$

The equality $R(\lambda, A)A = -E + \lambda R(\lambda, A)$ implies that

$$\|R(\lambda, A)A^m\|_{L(X)} \leqslant c_{25}\Delta_p(|\lambda|)$$

with

$$\Delta_p(t) \stackrel{\text{def}}{=} \begin{cases} t^{-1}, & [p]=0, \\ 1, & [p]>0, \end{cases} \qquad (t>0)$$

Therefore,

$$\|(\Theta(A_h,\alpha)A_h - \Theta(A,\alpha)\,A)\,A^m(A+\varepsilon E)^\mu v\|_X \leqslant c_{26}\|v\|_X\Delta_p(\alpha)h \quad \forall\alpha\in(0,\alpha_0] \tag{5.14}$$

In a similar way, for the second term in (5.13) with the use of Assumption 4.1 and Lemma 5.1 we obtain

$$\|(\Theta(A_h,\alpha)A_h - \Theta(A,\alpha)A)A^m[(A+\varepsilon E)^\mu - A^\mu]v\|_X$$
$$\leqslant c_{27}\|v\|_X\alpha^\mu\cdot\int_{\Gamma_\alpha}|1-\Theta(\lambda,\alpha)\lambda|\|(R(\lambda, A) - R(\lambda, A_h))A^m\|_{L(X)}\,|d\lambda| \leqslant c_{28}\|v\|_X\Delta_p(\alpha)\alpha^\mu h \tag{5.15}$$

Combining (5.8) and (5.10) – (5.15), we finally get

$$\left\|x_\alpha^{(h,\eta)} - x^*\right\|_X \leqslant c_{29}\Big(\frac{h+\eta}{\alpha} + (\Delta_p(\alpha)h + \alpha^p)\|v\|_X\Big) \tag{5.16}$$

The following statement is a direct consequence of the previous examinations.

Theorem 5.1. *Suppose Assumptions* 4.1 – – 4.3 *and* 5.1 *are satisfied. Let the initial discrepancy* $x^*-\xi$ *possess sourcewise representation* (4.10) *and the regularization parameter* $\alpha=\alpha(h,\eta)$ *be coordinated with the levels of errors* h, η *in such a way that* (5.6) *holds and*

$$\alpha(h,\eta)\in(0,\alpha_0], \quad \lim_{h,\eta\to 0}\alpha(h,\eta) = \lim_{h,\eta\to 0}\frac{h+\eta}{\alpha(h,\eta)} = 0 \tag{5.17}$$

Then (5.16) *and* (5.3) *hold.*

We conclude this section with a brief discussion of how one can relax condition (4.10), which seems to be the most restrictive among the hypotheses of Theorems 4.1 and 5.1 (along with Assumption 4.1). Assume that for a given approximate operator A_h there exist an element $v_h\in X$ and an exponent $p\in\mathbf{N}$ such that

$$x^*-\xi = A_h^p v_h + w_h, \quad \|w_h\|_X \leqslant \nu \tag{5.18}$$

The value ν in (5.18) is an estimate for the error in the sourcewise representation of the discrepancy $x^*-\xi$ with the use of the available operator A_h. Suppose ν is small enough along with h, η. By (5.2), we get

$$\|x_\alpha^{(h,\eta)} - x^*\|_X \leqslant c_{30}\|\Theta(A_h,\alpha)\|_{L(X)}(h+\eta) + \|(E-\Theta(A_h,\alpha)\,A_h)\,A_h^p v_h\|_X + \|E-\Theta(A_h,\alpha)\,A_h\|_{L(X)}\nu$$

Then, as in the proof of Theorem 5.1, we obtain

$$\|x_\alpha^{(h,\eta)} - x^*\|_X \leqslant c_{31}\Big(\frac{h+\eta}{\alpha} + \alpha^p\|v_h\|_X + \nu\Big) \tag{5.19}$$

Therefore, we have proved the following assertion.

Theorem 5.2. *Suppose Assumptions* 4.1 – 4.3 *and* 5.1 *are satisfied. Let the initial discrepancy* $x^*-\xi$ *possess the approximate sourcewise representation* (5.18) *with an exponent* $p\in\mathbf{N}$ *and the regularization parameter* $\alpha=\alpha(h,\eta)$ *be consistent with the levels of errors* h, η *such that* $\alpha\in(0,\alpha_0]$ *and* (5.6) *holds. Then, estimate* (5.19) *is valid.*

Corollary. *Suppose* (5.17) *and the hypotheses of Theorem* 5.2 *are satisfied. Assume that* $\sup_h\|v_h\|_X<\infty$. *Then,*

$$\overline{\lim_{h,\eta\to 0}}\|x_{\alpha(h,\eta)}^{(h,\eta)} - x^*\|_X \leqslant c_{31}\nu \tag{5.20}$$

Inequality (5.20) means that regularization algorithm (5.2) is stable with respect to errors in sourcewise representation (4.10).

As one more application of Theorem 5.2, let us prove the convergence of approximations $x^{(h,\eta)}_{\alpha(h,\eta)}$ defined in (5.2) without additional assumptions on the initial discrepancy. Suppose X is a reflexive Banach space. By [27, p. 637] and Assumption 4.1, the direct decomposition $X = \mathrm{cl}\, R(A) \oplus N(A)$ holds. Therefore, each element $x \in X$ possesses a unique representation in the form $x = y + z$ with $y \in \mathrm{cl}\, R(A)$ and $z \in N(A)$. Consequently, for each $\xi \in X$ there exists a unique element $x^* \in X^*$ such that $x^* - \xi \in \mathrm{cl}\, R(A)$ and for arbitrarily small $\varepsilon > 0$ there exist elements v_ε and w_ε such that $x^* - \xi = Av_\varepsilon + w_\varepsilon$, $\|w_\varepsilon\|_X < \varepsilon$. Taking in (5.19) $\nu = h\|v_\varepsilon\|_X + \varepsilon$, we obtain

$$\left\|x_\alpha^{(h,\eta)} - x^*\right\|_X \leqslant c_{31}\Big(\frac{h+\eta}{\alpha} + (\alpha + h)\|v_\varepsilon\|_X + \varepsilon\Big)$$

If we combine this inequality with (5.17), we get $\overline{\lim\limits_{h,\eta\to 0}}\|x^{(h,\eta)}_{\alpha(h,\eta)} - x^*\|_X \leqslant c_{31}\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we come to the following result.

Theorem 5.3. *Let X be a reflexive Banach space, Assumptions* 4.1 – 4.3 *and* 5.1 *and condition* (4.14) *with $p = 1$ be satisfied. Then, approximations $x^{(h,\eta)}_{\alpha(h,\eta)}$ generated by* (5.2) *and an element $x^* \in X$ such that $x^* - \xi \in \mathrm{cl}\, R(A)$ satisfy* (5.3).

6. On the necessity of sourcewise representation conditions. In this section, we consider the question of whether condition (4.10) is necessary for the power rate of convergence (4.17). For simplicity, let $q = 0$. We shall prove that representation (4.10), which is sufficient for (4.17), actually appears to be very close to a necessary one, so that under appropriate additional conditions on the generating functions $\Theta(\lambda, \alpha)$ estimate (4.17) implies

$$x^* - \xi \in R(A^{p-\varepsilon}) \quad \forall \varepsilon \in (0, p) \tag{6.1}$$

(compare with Theorems 3.1 and 3.2 for $q = 0$).

From the technical point of view, it will be convenient to assume that the generating functions $\Theta(\lambda, \alpha)$ are defined for all positive values of the regularization parameter $\alpha \in (0, \infty)$. Assumption 4.2, then, implies that the operator

$$\Theta(A, \alpha) = \frac{1}{2\pi i}\int\limits_{\gamma_\alpha} \Theta(\lambda, \alpha) R(\lambda, A)\, d\lambda \tag{6.2}$$

is defined for all $\alpha \in (0, \infty)$. By (4.3) and (6.2) and with the use of the equality $Ax^* = f$, we obtain

$$x_\alpha - x^* = (E - \Theta(A, \alpha)A)\,(\xi - x^*) = \frac{1}{2\pi i}\int\limits_{\gamma_\alpha} (1 - \Theta(\lambda, \alpha)\lambda)\, R(\lambda, A)\,(\xi - x^*)\, d\lambda \tag{6.3}$$

Suppose the following Assumptions 6.1 and 6.2 are satisfied.

Assumption 6.1. For all $r \in (0, r_0]$

$$\sup_{\alpha \in [\alpha_0, \infty)} \alpha^{-r} \int\limits_{\gamma_\alpha} \frac{|1 - \Theta(\lambda, \alpha)\lambda|}{|\lambda|}\, |d\lambda| < \infty \tag{6.4}$$

Inequality (6.4) is a weakened variant of Assumption 4.3 with $p = 0$ written for $\alpha \in [\alpha_0, \infty)$. We denote

$$D(R_0, d_0, \varphi_0) = \{(\lambda, \alpha) \colon \lambda \in K_\alpha(R_0, d_0, \varphi_0), \alpha \in (0, \infty)\}$$

Assumption 6.2. There exists $\varepsilon_0 > 0$ such that the function $\Theta(\lambda, \alpha)$ is continuous in (λ, α) on the set $D(R_0 + \varepsilon_0, d_0, \varphi_0)$.

Assume that there exists a constant $l_0 > 0$ such that

$$\|x_\alpha - x^*\|_X \leqslant l_0 \alpha^p (p > 0) \quad \forall \alpha \in (0, \alpha_0] \tag{6.5}$$

From Assumption 6.2 and representation (6.2) we conclude that the element x_α depends on $\alpha \in (0, \infty)$

continuously. Besides, by (4.5) and (6.3) – (6.5), for all κ, $0<\kappa<\min\{\frac{2}{3}p, 2r_0\}$ we have

$$\int\limits_0^\infty \alpha^{-p-1+\kappa}\|x_\alpha-x^*\|_X\,d\alpha=\int\limits_0^{\alpha_0}\alpha^{-p-1+\kappa}\|x_\alpha-x^*\|_X\,d\alpha+\int\limits_{\alpha_0}^\infty \alpha^{-p-1+\kappa}\|x_\alpha-x^*\|_X\,d\alpha\leqslant$$

$$\leqslant l_0\int\limits_0^{\alpha_0}\alpha^{-1+\kappa}\,d\alpha+\int\limits_{\alpha_0}^\infty \alpha^{-p+1+\kappa}\|E-\Theta(A,\alpha)A\|_{L(X)}\|x^*-\xi\|_X\,d\alpha \tag{6.6}$$

$$\leqslant l_0\int\limits_0^{\alpha_0}\alpha^{-1+\kappa}\,d\alpha+\frac{c_{10}}{2\pi}\|x^*-\xi\|_X\cdot\int\limits_{\alpha_0}^\infty \alpha^{-p-1+3\kappa/2}\Big(\alpha^{-\kappa/2}\int\limits_{\gamma_\alpha}\frac{|1-\Theta(\lambda,\alpha)\lambda|}{|\lambda|}\,|d\lambda|\Big)\,d\alpha<\infty$$

Therefore, the formula

$$w_\kappa=\int\limits_0^\infty \alpha^{-p-1+\kappa}(x^*-x_\alpha)\,d\alpha \tag{6.7}$$

defines an element $w_\kappa\in X$ (here the integral exists in Bochner's sense).

The plan of our further examinations looks like as follows. Apart from the operator A, we shall consider its regularization $A_\varepsilon=A+\varepsilon E$, $\varepsilon>0$, and define the element

$$u_\kappa^{(\varepsilon)}=\int\limits_0^\infty \alpha^{-p-1+\kappa}\left(\Theta(A,\alpha)A-\Theta(A_\varepsilon,\alpha)A_\varepsilon\right)(x^*-\xi)\,d\alpha \tag{6.8}$$

We shall prove the convergence of the integral in (6.8) and establish an upper estimate for the norm $\|u_\kappa^{(\varepsilon)}\|_X$. This will imply the convergence of the integral

$$\int\limits_0^\infty \alpha^{-p-1+\kappa}\left(E-\Theta(A_\varepsilon,\alpha)A_\varepsilon\right)(x^*-\xi)\,d\alpha=u_\kappa^{(\varepsilon)}+w_\kappa\overset{\text{def}}{=}w_\kappa^{(\varepsilon)} \tag{6.9}$$

with the estimate $\|w_\kappa^{(\varepsilon)}-w_\kappa\|_X=\|u_\kappa^{(\varepsilon)}\|_X$. At the final stage, by direct calculation we shall establish the equality

$$A_\varepsilon^{p-\kappa}w_\kappa^{(\varepsilon)}=C(p,\kappa)(x^*-\xi) \tag{6.10}$$

with a constant $C(p,\kappa)>0$. On the other hand, with the use of the above-mentioned estimate we shall prove that $A_\varepsilon^{p-\kappa}w_\kappa^{(\varepsilon)}=A^{p-\kappa}w_\kappa$ for a sufficiently small $\varepsilon=\varepsilon_n\to 0$. This will give us the desired representation (6.1) immediately.

Suppose the next assumption is satisfied.

Assumption 6.3. For all $\alpha\in(0,\infty)$,

$$1-\Theta(\lambda,\alpha)\lambda\neq 0\quad\forall\lambda\in K_\alpha(R_0,d_0,\varphi_0)$$

By the spectral mapping theorem [28, p. 220] and Assumption 6.3, the operator $E-\Theta(A,\alpha)A$ possesses a continuous inverse for all $\alpha\in(0,\infty)$. Therefore,

$$u_\kappa^{(\varepsilon)}=\int\limits_0^\infty \alpha^{-p-1+\kappa}\,\psi(A,\alpha,\varepsilon)\left(E-\Theta(A,\alpha)\,A\right)(x^*-\xi)\,d\alpha \tag{6.11}$$

where

$$\psi(\lambda,\alpha,\varepsilon)\overset{\text{def}}{=}\frac{\Theta(\lambda,\alpha)\lambda-\Theta(\lambda+\varepsilon,\alpha)(\lambda+\varepsilon)}{1-\Theta(\lambda,\alpha)\lambda}$$

From Assumptions 6.2 and 6.3 it follows that for each $\varepsilon\in(0,\varepsilon_0]$ the function $\psi(\lambda,\alpha,\varepsilon)$ is continuous in (λ,α) on $D(R_0,d_0,\varphi_0)$. Therefore, the operator $\psi(A,\alpha,\varepsilon)$ depends on α continuously with respect to the norm of $L(X)$ for $\alpha\in(0,\infty)$. To prove the convergence of Bochner's integral in (6.11), it suffices to establish the existence of the Lebesgue integral in the right-hand side of the inequality

$$\int\limits_0^\infty \alpha^{-p-1+\kappa}\,\|\psi(A,\alpha,\varepsilon)\left(E-\Theta(A,\alpha)\,A\right)(x^*-\xi)\|_X\,d\alpha\leqslant\int\limits_0^\infty \alpha^{-p-1+\kappa}\,\|\psi(A,\alpha,\varepsilon)\|_{L(X)}\,\|x_\alpha-x^*\|_X\,d\alpha \tag{6.12}$$

By (4.5),

$$\|\psi(A,\alpha,\varepsilon)\|_{L(X)} \leqslant \frac{1}{2\pi}\int\limits_{\gamma_\alpha} |\psi(\lambda,\alpha,\varepsilon)|\,\|R(\lambda,A)\|_{L(X)}\,|d\lambda| \leqslant \frac{c_{10}}{2\pi}\int\limits_{\gamma_\alpha} \frac{|\psi(\lambda,\alpha,\varepsilon)|}{|\lambda|}|\,d\lambda| \tag{6.13}$$

Let us supplement the previous conditions on $\Theta(\lambda,\alpha)$ by the following one.

Assumption 6.4. There exist constants $\varepsilon_1 \in (0,\varepsilon_0]$ and $s_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1]$ the following inequality holds:

$$\int\limits_{\gamma_\alpha} \frac{|\psi(\lambda,\alpha,\varepsilon)|}{|\lambda|}|\,d\lambda| \overset{\text{def}}{=} M(\alpha,\varepsilon) < \infty \tag{6.14}$$

Moreover, for all $s \in (0,s_0]$

$$\sup_{\alpha\in(0,\alpha_0]}(\alpha^s M(\alpha,\varepsilon)) + \sup_{\alpha\in[\alpha_0,\infty)}(\alpha^{-s}M(\alpha,\varepsilon)) < \infty \tag{6.15}$$

and

$$\sup_{\alpha\in(0,\alpha_0]}\Big(\alpha^s\int\limits_0^{\varepsilon_1}\frac{M(\alpha,\varepsilon)}{\varepsilon}\,d\varepsilon\Big) + \sup_{\alpha\in[\alpha_0,\infty)}\Big(\alpha^{-s}\int\limits_0^{\varepsilon_1}\frac{M(\alpha,\varepsilon)}{\varepsilon}\,d\varepsilon\Big) < \infty \tag{6.16}$$

Estimates (6.4)–(6.6) and (6.12)–(6.15) prove that the integral in (6.11) exists for each $\varepsilon \in (0,\varepsilon_1]$ and $0 < \kappa < \min\{\frac{1}{2}p, 2r_0, 2s_0\}$, since

$$\int\limits_0^\infty \alpha^{-p-1+\kappa}\|\psi(A,\alpha,\varepsilon)\|_{L(X)}\|x_\alpha - x^*\|_X\,d\alpha \leqslant$$

$$\leqslant \frac{c_{10}}{2\pi}\Big(\int\limits_0^{\alpha_0}\alpha^{-p-1+\kappa}M(\alpha,\varepsilon)\|x_\alpha - x^*\|_X\,d\alpha + \int\limits_{\alpha_0}^\infty \alpha^{-p-1+\kappa}M(\alpha,\varepsilon)\|x_\alpha - x^*\|_X\,d\alpha\Big)$$

$$\leqslant \frac{c_{10}}{2\pi}\Big(l_0\int\limits_0^{\alpha_0}\alpha^{-1+\kappa/2}\left(\alpha^{\kappa/2}M(\alpha,\varepsilon)\right)d\alpha + \int\limits_{\alpha_0}^\infty \alpha^{-p-1+3\kappa/2}\left(\alpha^{-\kappa/2}M(\alpha,\varepsilon)\right)\|x_\alpha - x^*\|_X\,d\alpha\Big) < \infty$$

Therefore, for each $\varepsilon \in (0,\varepsilon_1]$ the integral in (6.9) is well defined and represents an element $w_\kappa^{(\varepsilon)} \in X$. By (6.11), (6.13), and (6.14) and with the use of the Fubini theorem [19, p. 318], we obtain

$$\int\limits_0^{\varepsilon_1}\frac{\|w_\kappa^{(\varepsilon)} - w_\kappa\|_X}{\varepsilon}\,d\varepsilon = \int\limits_0^{\varepsilon_1}\alpha^{-p-1+\kappa}\,\frac{\|\psi(A,\alpha,\varepsilon)\|_{L(X)}}{\varepsilon}\,\|x_\alpha - x^*\|_X\,d\alpha \leqslant$$

$$\leqslant \frac{c_{10}}{2\pi}\Bigg(\int\limits_0^{\alpha_0}\alpha^{-p-1+\kappa/2}\|x_\alpha - x^*\|_X\cdot\Big(\alpha^{\kappa/2}\int\limits_0^{\varepsilon_1}\frac{M(\alpha,\varepsilon)}{\varepsilon}\,d\varepsilon\Big)d\alpha$$

$$+\int\limits_{\alpha_0}^\infty \alpha^{-p-1+3\kappa/2}\|x_\alpha - x^*\|_X\cdot\Big(\alpha^{-\kappa/2}\int\limits_0^{\varepsilon_1}\frac{M(\alpha,\varepsilon)}{\varepsilon}\,d\varepsilon\Big)d\alpha\Bigg), \quad 0 < \kappa < \min\left\{\frac{1}{2}p, 2r_0, 2s_0\right\}$$

Now, by (6.5), (6.6), and (6.16),

$$\int\limits_0^{\varepsilon_1}\frac{\|w_\kappa^{(\varepsilon)} - w_\kappa\|_X}{\varepsilon}\,d\varepsilon < \infty \tag{6.17}$$

Inequality (6.17) implies that there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$, such that $\lim\limits_{n\to\infty}\|w_\kappa^{(\varepsilon_n)} - w_\kappa\|_X = 0$. In fact, suppose there exists $c_{32} > 0$ such that

$$\left\|w_\kappa^{(\varepsilon)} - w_\kappa\right\|_X \geqslant c_{32} \quad \forall\varepsilon \in (0,\varepsilon_2], \quad \varepsilon_2 \leqslant \varepsilon_1$$

Then contrary to (6.17), we get

$$\int\limits_0^{\varepsilon_1}\frac{\|w_\kappa^{(\varepsilon)} - w_\kappa\|_X}{\varepsilon}\,d\varepsilon \geqslant \int\limits_0^{\varepsilon_2}\frac{c_{32}}{\varepsilon}\,d\varepsilon = \infty$$

Let $\kappa > 0$ be small enough. Choose arbitrarily $m \in \mathbf{N}$ such that $p-\kappa \in (m, m+1)$. By Definition 4.1,

$$A_\varepsilon^{p-\kappa} = \frac{(-1)^m \sin \pi(p-\kappa)}{\pi} \int_0^\infty \int_0^\infty t^{p-\kappa-m-1}(tE + A_\varepsilon)^{-1} A_\varepsilon^{m+1}\, dt$$

Therefore, from (6.9)

$$A_\varepsilon^{p-\kappa} w_\kappa^{(\varepsilon)} = \frac{(-1)^m \sin \pi(p-\kappa)}{\pi} \int_0^\infty \int_0^\infty t^{p-\kappa-m-1} \alpha^{-p-1+\kappa} (tE + A_\varepsilon)^{-1} A_\varepsilon^{m+1} (E - \Theta(A_\varepsilon, \alpha) A_\varepsilon)(x^* - \xi)\, d\alpha\, dt \tag{6.18}$$

Given r, r_1, $r_2 > 0$ and z, z_1, $z_2 \in \mathbf{C}$ such that $r_1 \leqslant r_2$, $z \neq 0$, $\arg z_1 \leqslant \arg z_2$, we define the contours

$$\Gamma_r(z_1, z_2) = \{\zeta \in \mathbf{C} \colon |\zeta| = r, \arg z_1 \leqslant \arg \zeta \leqslant \arg z_2\}$$

$$\Gamma_{(r_1,r_2)}(z) = \{\zeta \in \mathbf{C} \colon r_1 \leqslant |\zeta| \leqslant r_2, \arg \zeta = \arg z\}$$

Let

$$\eta(\lambda, t, \alpha) = (t+\lambda)^{-1} \lambda^{m+1} (1 - \Theta(\lambda, \alpha)\lambda)$$

Note that the operator $\eta(A_\varepsilon, t, \alpha)$ admits representation (4.2) with the positively oriented contour

$$\Gamma = \Gamma^{(\varepsilon)} \overset{\text{def}}{=} \Gamma_{\frac{\varepsilon}{2}}(e^{-i\varphi_0}, e^{i\varphi_0}) \cup \Gamma_{R_0}(e^{-i\varphi_0}, e^{i\varphi_0}) \cup \Gamma_{(\frac{\varepsilon}{2}, R_0)}(e^{i\varphi_0}) \cup \Gamma_{(\frac{\varepsilon}{2}, R_0)}(e^{-i\varphi_0})$$

Indeed, it can easily be checked that the contour $\Gamma^{(\varepsilon)}$ surrounds the spectrum $\sigma(A_\varepsilon) = \{\lambda + \varepsilon \colon \lambda \in \sigma(A)\}$ and for all $t, \alpha \in (0, \infty)$ lies in the domain where the function $\eta(\lambda, t, \alpha)$ is analytic in λ. By (4.2) and (6.18), we obtain the representation

$$A_\varepsilon^{p-\kappa} w_\kappa^{(\varepsilon)} = D(p,\kappa) \int_0^\infty \int_0^\infty \int_{\Gamma^{(\varepsilon)}} \alpha^{-p-1+\kappa} t^{p-\kappa-m-1} (t+\lambda)^{-1} \lambda^{m+1} (1 - \Theta(\lambda, \alpha)\lambda) R(\lambda, A_\varepsilon)(x^* - \xi)\, d\lambda\, d\alpha\, dt \tag{6.19}$$

with $D(p,\kappa) = \frac{(-1)^m \sin \pi(p-\kappa)}{2\pi^2 i}$. To transform (6.19) with the use of the Fubini theorem (see [29, p. 354]), we need to prove that

$$J \overset{\text{def}}{=} \int_{\Gamma^{(\varepsilon)}} \Big(\int_0^\infty \int_0^\infty \alpha^{-p-1+\kappa} t^{p-\kappa-m-1} |t+\lambda|^{-1} |\lambda|^{m+1} |1 - \Theta(\lambda,\alpha)\lambda| \|R(\lambda, A_\varepsilon)(x^* - \xi)\|_X\, d\alpha\, dt \Big) |d\lambda| < \infty \tag{6.20}$$

Since for each $\varepsilon \in (0, \varepsilon_1]$

$$\sup_{\lambda \in \Gamma^{(\varepsilon)}} \|R(\lambda, A_\varepsilon)\,(x^* - \xi)\|_X \overset{\text{def}}{=} E(\varepsilon) < \infty$$

the following estimate is valid:

$$J \leqslant E(\varepsilon) \cdot \int_{\Gamma^{(\varepsilon)}} |\lambda|^{m+1} \Big(\int_0^\infty \alpha^{-p-1+\kappa} |1 - \Theta(\lambda,\alpha)\lambda|\, d\alpha \Big) \Big(\int_0^\infty t^{p-\kappa-m-1} |t+\lambda|^{-1} dt \Big) |d\lambda| \tag{6.21}$$

Suppose the functions $\Theta(\lambda, \alpha)$ satisfy the following additional condition.

Assumption 6.5. The function $g(\zeta) \overset{\text{def}}{=} 1 - \Theta(\lambda, \lambda\zeta)\,\lambda$ does not depend on λ when $\lambda \in K(\varphi_0) \backslash \{0\}$. Besides, $g(\zeta) \overset{\text{def}}{=} 1 - \Theta(\lambda, \lambda\zeta)\,\lambda$ is analytic on $D_0 \supset K(\varphi_0) \backslash \{0\}$ and for all $t \in (0, p)$

$$\sup_{|\varphi| \leqslant \varphi_0} \int_0^\infty \tau^{-t-1} |g(e^{i\varphi}\tau)|\, d\tau \overset{\text{def}}{=} N(t) < \infty \tag{6.22}$$

$$\lim_{r \to 0+} r^{-t-1} \int_{\Gamma_r(e^{-i\varphi_0}, e^{i\varphi_0})} |g(\zeta)|\, |d\zeta| = 0 \tag{6.23}$$

$$\lim_{R\to\infty} R^{-t-1} \int\limits_{\Gamma_R(e^{-i\varphi_0},e^{i\varphi_0})} |g(\zeta)|\,|d\zeta| = 0 \tag{6.24}$$

Let us analyze each of the internal integrals in (6.21). Using the change of variables $\alpha = |\lambda|\tau$, $t = |\lambda|\tau$ and inequality (6.22), we get the estimates

$$\int\limits_0^\infty \alpha^{-p-1+\kappa}|1-\Theta(\lambda,\alpha)\lambda|\,d\alpha = |\lambda|^{-p+\kappa}\int\limits_0^\infty \tau^{-p-1+\kappa}|g(e^{-i\arg\lambda}\tau)|\,d\tau \leqslant N(p-\kappa)|\lambda|^{-p+\kappa} \tag{6.25}$$

$$\int\limits_0^\infty t^{p-\kappa-m-1}|t+\lambda|^{-1}dt = |\lambda|^{p-\kappa-m-1}\int\limits_0^\infty \tau^{p-\kappa-m-1}|\tau+|\lambda|^{-1}\lambda|^{-1}d\tau \leqslant P(p,\kappa)|\lambda|^{p-\kappa-m-1} \quad \forall\lambda\in\Gamma^{(\varepsilon)} \tag{6.26}$$

where $P(p,\kappa)$ is an absolute constant that does not depend on λ, $0<\kappa<\min\{\frac{1}{2}p, 2r_0, 2s_0, t_0\}$. The desired relation (6.20) follows from (6.21), (6.25), and (6.26) immediately. Changing the order of integrals in (6.19), we obtain

$$A_\varepsilon^{p-\kappa} w_\kappa^{(\varepsilon)} = D(p,\kappa)\int\limits_{\Gamma^{(\varepsilon)}} \lambda^{m+1}\Big(\int\limits_0^\infty \alpha^{-p-1+\kappa}(1-\Theta(\lambda,\alpha)\lambda)\,d\alpha\Big)\Big(\int\limits_0^\infty t^{p-\kappa-m-1}(t+\lambda)^{-1}dt\Big)R(\lambda,A_\varepsilon)(x^*-\xi)\,d\lambda \tag{6.27}$$

Given $z\in\mathbf{C}$, denote $\Lambda(z) \overset{\text{def}}{=} \{\zeta\in\mathbf{C}\colon \zeta = tz,\ t\geqslant 0\}$. As usual, $\bar z$ is the complex conjugate for $z\in\mathbf{C}$. The change of variables $\alpha=\lambda\zeta$ allows us to transform the first of the internal integrals in (6.27) as

$$\int\limits_0^\infty \alpha^{-p-1+\kappa}(1-\Theta(\lambda,\alpha)\,\lambda)\,d\alpha = \lambda^{-p+\kappa}\int\limits_{\Lambda(\bar\lambda)} \zeta^{-p-1+\kappa}g(\zeta)\,d\zeta \tag{6.28}$$

where the integration over $\Lambda(\bar\lambda)$ is performed from $\zeta=0$ to $\zeta=\infty$. We claim that the value $G(p,\kappa)$ of the integral in the right-hand side of (6.28) does not depend on $\lambda\in\Gamma^{(\varepsilon)}$. Indeed, choose $\lambda_1,\lambda_2\in\Gamma^{(\varepsilon)}$ such that $\arg\bar\lambda_1 < \arg\bar\lambda_2$ and construct the positively oriented contour

$$\Gamma_{(r,R)}(\bar\lambda_1,\bar\lambda_2) = \Gamma_r(\bar\lambda_1,\bar\lambda_2)\cup\Gamma_R(\bar\lambda_1,\bar\lambda_2)\cup\Gamma_{(r,R)}(\bar\lambda_1)\cup\Gamma_{(r,R)}(\bar\lambda_2)$$

with $0<r<R$. Since $g(\zeta)$ is analytic on $K(\varphi_0)\backslash\{0\}\supset\Gamma_{(r,R)}(\bar\lambda_1,\bar\lambda_2)$, we have

$$\int\limits_{\Gamma_{(r,R)}(\bar\lambda_1,\bar\lambda_2)} \zeta^{-p-1+\kappa}g(\zeta)\,d\zeta = \int\limits_{\Gamma_r(\bar\lambda_1,\bar\lambda_2)} \zeta^{-p-1+\kappa}g(\zeta)\,d\zeta + \int\limits_{\Gamma_R(\bar\lambda_1,\bar\lambda_2)} + \int\limits_{\Gamma_{(r,R)}(\bar\lambda_1)} + \int\limits_{\Gamma_{(r,R)}(\bar\lambda_2)}$$

Passing to limits as $r\to 0$, $R\to\infty$ and using (6.23) and (6.24), we obtain

$$\int\limits_{\Lambda(\bar\lambda_1)} \zeta^{-p-1+\kappa}g(\zeta)\,d\zeta = \int\limits_{\Lambda(\bar\lambda_2)} \zeta^{-p-1+\kappa}g(\zeta)\,d\zeta$$

In a similar way, for the second of the internal integrals in (6.27) we get

$$\int\limits_0^\infty t^{p-\kappa-m-1}(t+\lambda)^{-1}dt = \lambda^{p-\kappa-m-1}\int\limits_{\Lambda(\bar\lambda)} \zeta^{p-\kappa-m-1}(1+\zeta)^{-1}d\zeta \tag{6.29}$$

Note that the value $H(p,\kappa)$ of the integral in the right-hand side of (6.29) does not depend on $\lambda\in\Gamma^{(\varepsilon)}$. By (6.27) – (6.29), for each $\varepsilon\in(0,\varepsilon_1]$ we have

$$A_\varepsilon^{p-\kappa}w_\kappa^{(\varepsilon)} = D(p,\kappa)\,G(p,\kappa)\,H(p,\kappa)\int\limits_{\Gamma^{(\varepsilon)}} R(\lambda,A_\varepsilon)(x^*-\xi)\,d\lambda = C(p,\kappa)(x^*-\xi)$$

where $C(p,\kappa) = 2\pi i D(p,\kappa)G(p,\kappa)H(p,\kappa)\neq 0$. Therefore, we have proved equality (6.10).

The main result of this section is the following.

Theorem 6.1. *Assume that approximations x_α satisfy* (6.5). *Let Assumptions* 4.1 – 4.3 *and* 6.2 – 6.5 *be valid. Then, for each $\varepsilon \in (0,p)$ the initial discrepancy $x^* - \xi$ possesses the sourcewise representation $x^* - \xi \in R(A^{p-\varepsilon})$.*

Proof. We claim that $A_{\varepsilon_n}^{p-\kappa} w_\kappa^{(\varepsilon_n)} = A^{p-\kappa} w_\kappa$ with w_κ and $\{\varepsilon_n\}$ defined above, $\varepsilon_n > 0$, $\lim\limits_{n\to\infty} \varepsilon_n = 0$, $\lim\limits_{n\to\infty} \|w_\kappa^{(\varepsilon_n)} - w_\kappa\|_X = 0$. Let $m = [p-\kappa]$, $\mu = p - \kappa - m$ be such that $p - \kappa = m + \mu$, $\mu \in (0,1)$. We see that

$$\begin{aligned} \|A_{\varepsilon_n}^{p-\kappa} w_\kappa^{(\varepsilon_n)} - A^{p-\kappa} w_\kappa\|_X &= \|A_{\varepsilon_n}^{m+\mu} w_\kappa^{(\varepsilon_n)} - A^{m+\mu} w_\kappa\|_X \\ &\leqslant \|A_{\varepsilon_n}^{m+\mu} - A^{m+\mu}\|_{L(X)} \|w_\kappa^{(\varepsilon_n)}\|_X + \|A^{m+\mu}\|_{L(X)} \|w_\kappa^{(\varepsilon_n)} - w_\kappa\|_X \end{aligned} \tag{6.30}$$

It can easily be checked that

$$\|A_{\varepsilon_n}^{m+\mu} - A^{m+\mu}\|_{L(X)} \leqslant \|A_{\varepsilon_n}^{m} - A^{m}\|_{L(X)} \|A^\mu\|_{L(X)} + \|A_{\varepsilon_n}\|_{L(X)}^m \|A_{\varepsilon_n}^\mu - A^\mu\|_{L(X)} \tag{6.31}$$

and

$$\|A_\varepsilon^m - A^m\|_{L(X)} \leqslant c_{33}\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_1] \tag{6.32}$$

From Lemma 4.1 it follows

$$\|A_\varepsilon^\mu - A^\mu\|_{L(X)} \leqslant c_{11}\varepsilon^\mu \quad \forall \varepsilon \in (0, \varepsilon_1] \tag{6.33}$$

Note that the constants c_{11} and c_{33} in (6.32) and (6.33) depend on A, p, and κ only. Combining (6.30) – (6.33), we obtain

$$\lim_{n\to\infty} \left\| A_{\varepsilon_n}^{p-\kappa} w_\kappa^{(\varepsilon_n)} - A^{p-\kappa} w_\kappa \right\|_X = 0 \tag{6.34}$$

From (6.10) we see that the element $A_\varepsilon^{p-\kappa} w_\kappa^{(\varepsilon)}$ does not depend on ε. Therefore, (6.34) implies $A^{p-\kappa} w_\kappa = C(p,\kappa)(x^* - \xi)$. Consequently, for an element $v_\kappa = C(p,\kappa)^{-1} w_\kappa$ we have

$$A^{p-\kappa} v_\kappa = x^* - \xi$$

and for a sufficiently small $\kappa > 0$ we have $x^* - \xi \in R(A^{p-\kappa})$. Since $A^{p_1+p_2} = A^{p_1} A^{p_2}$, p_1, $p_2 > 0$, for each $\varepsilon \in (0,p)$ we get $x^* - \xi \in R(A^{p-\varepsilon})$. This completes the proof.

In conclusion, we consider some examples of procedures (4.3) that satisfy the assumptions listed above.

Example 6.1. Function (2.17) satisfies Assumption 4.2 with $D_\alpha = \mathbf{C}\backslash\{-\alpha\}$. Assumptions 4.3 and 5.1 can easily be verified with the family of contours [4, p. 53]

$$\Gamma_\alpha = \mathrm{fr}(S_{R_0}(0) \backslash S_{(1-d_0)\alpha}(-\alpha)), \quad \alpha > 0 \tag{6.35}$$

Simple calculations prove that inequality (4.14) holds for all $p_0 \in (0,1]$. Assumptions 6.1 – 6.5 are also satisfied.

Example 6.2. Assumption 4.2 is valid for function (2.18) with $D_\alpha = \mathbf{C}\backslash\{-\alpha\}$. Direct calculations prove that Assumptions 4.3 and 5.1 are satisfied for all $p_0 \in (0, N]$ if the contour Γ_α is constructed as in (6.35). Verification of Assumption 6.1 – 6.5 is immediate.

Example 6.3. Function (2.22) is analytic on the entire complex plane $\mathbf{C}$. In this case, it is convenient to choose $\Gamma_\alpha = \gamma_\alpha$, $\alpha > 0$. Examinations similar to those from [4, pp. 51 – 53] show that Assumption 4.3 is satisfied for each $p_0 > 0$ provided that Assumption 4.1 is valid with $\varphi_0 \in (0, \frac{\pi}{2})$. Assumption 5.1 and 6.1 – 6.5 are also satisfied.

Example 6.4. Function (2.25) with $g(\lambda) \equiv \mu_0$ is analytic everywhere in $\mathbf{C}$. Suppose Assumption 4.1 is satisfied with some $\varphi_0 \in (0, \frac{\pi}{4})$. In addition assume that $0 < \mu_0 < \|A\|_{L(X)}^{-1} \sqrt{2\cos 2\varphi_0}$. As in Example 6.3, we set $\Gamma_\alpha = \gamma_\alpha, \alpha > 0$. Following the scheme of arguing from [4, pp. 51–52] one can verify that Assumptions 4.3 and 5.1 are valid for each $p_0 > 0$.

Remark 6.1. Procedures (4.3) and (5.2) for the functions $\Theta(\lambda, \alpha)$ from Examples 6.1 – 6.4 can practically be implemented as in Examples 2.1, 2.2, 2.4, and 2.5.

Remark 6.2. Since Assumption 4.3 in Examples 6.3 and 6.4 is valid without upper bounds on p_0, the corresponding procedures (4.3) are free of the saturation phenomenon.

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